

MISCELLANEOUS MATHEMATICAL CONSTANTS

π CONSTANTS

$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ 41971\ 69399\ 37511$
 $1/\pi = 0.31830\ 98861\ 83790\ 67153\ 77675\ 26745\ 02872\ 40689\ 19291\ 48091$
 $\pi^2 = 9.86960\ 44010\ 89358\ 61883\ 44909\ 99876\ 15113\ 53136\ 99407\ 24079$
 $\log_e \pi = 1.14472\ 98858\ 49400\ 17414\ 34273\ 51353\ 05871\ 16472\ 94812\ 91531$
 $\log_{10} \pi = 0.49714\ 98726\ 94133\ 85435\ 12682\ 88290\ 89887\ 36516\ 78324\ 38044$
 $\log_{10} \sqrt{2\pi} = 0.39908\ 99341\ 79057\ 52478\ 25035\ 91507\ 69595\ 02099\ 34102\ 92128$

CONSTANTS INVOLVING e

$e = 2.71828\ 18284\ 59045\ 23536\ 02874\ 71352\ 66249\ 77572\ 47093\ 69996$
 $1/e = 0.36787\ 94411\ 71442\ 32159\ 55237\ 70161\ 46086\ 74458\ 11131\ 03177$
 $e^2 = 7.38905\ 60989\ 30650\ 22723\ 04274\ 60575\ 00781\ 31803\ 15570\ 55185$
 $M = \log_{10} e = 0.43429\ 44819\ 03251\ 82765\ 11289\ 18916\ 60508\ 22943\ 97005\ 80367$
 $1/M = \log_e 10 = 2.30258\ 50929\ 94045\ 68401\ 79914\ 54684\ 36420\ 76011\ 01488\ 62877$
 $\log_{10} M = 9.63778\ 43113\ 00536\ 78912\ 29674\ 98645\ -10$

π^e AND e^π CONSTANTS

$\pi^e = 22.45915\ 77183\ 61045\ 47342\ 71522$
 $e^\pi = 23.14069\ 26327\ 79269\ 00572\ 90864$
 $e^{-\pi} = 0.04321\ 39182\ 63772\ 24977\ 44177$
 $e^{1/2\pi} = 4.81047\ 73809\ 65351\ 65547\ 30357$
 $i^i = e^{-1/2\pi} = 0.20787\ 95763\ 50761\ 90854\ 69556$

NUMERICAL CONSTANTS

$\sqrt{2} = 1.41421\ 35623\ 73095\ 04880\ 16887\ 24209\ 69807\ 85696\ 71875\ 37695$
 $\sqrt[3]{2} = 1.25992\ 10498\ 94873\ 16476\ 72106\ 07278\ 22835\ 05702\ 51464\ 70151$
 $\log_e 2 = 0.69314\ 71805\ 59945\ 30941\ 72321\ 21458\ 17656\ 80755\ 00134\ 36026$
 $\log_{10} 2 = 0.30102\ 99956\ 63981\ 19521\ 37388\ 94724\ 49302\ 67881\ 89881\ 46211$
 $\sqrt{3} = 1.73205\ 08075\ 68877\ 29352\ 74463\ 41505\ 87236\ 69428\ 05253\ 81039$
 $\sqrt[3]{3} = 1.44224\ 95703\ 07408\ 38232\ 16383\ 10780\ 10958\ 83918\ 69253\ 49935$
 $\log_e 3 = 1.09861\ 22886\ 68109\ 69139\ 52452\ 36922\ 52570\ 46474\ 90557\ 82275$
 $\log_{10} 3 = 0.47712\ 12547\ 19662\ 43729\ 50279\ 03255\ 11530\ 92001\ 28864\ 19070$

OTHER CONSTANTS

Euler's Constant $\gamma = 0.57721\ 56649\ 01532\ 86061$
 $\log_e \gamma = -0.54953\ 93129\ 81644\ 82234$
Golden Ratio $\phi = 1.61803\ 39887\ 49894\ 84820\ 45868\ 34365\ 63811\ 77203\ 09180$

EXPONENTIAL AND HYPERBOLIC FUNCTIONS AND THEIR COMMON LOGARITHMS

x	e ^x		e ^{-x} (value)	sinh x		cosh x		tanh x (value)
	Value	log ¹⁰		Value	log ¹⁰	value	log ¹⁰	
0.00	1.0000	0.00000	1.00000	0.0000	— ∞	1.0000	0.00000	0.00000
0.01	1.0101	.00434	0.99005	.0100	2̄.00001	1.0001	.00002	.01000
0.02	1.0202	.00869	.98020	.0200	2̄.30106	1.0002	.00009	.02000
0.03	1.0305	.01303	.97045	.0300	2̄.47719	1.0005	.00020	.02999
0.04	1.0408	.01737	.96079	.0400	2̄.60218	1.0008	.00035	.03998
0.05	1.0513	.02171	.95123	.0500	2̄.69915	1.0013	.00054	.04996
0.06	1.0618	.02606	.94176	.0600	2̄.77841	1.0018	.00078	.05993
0.07	1.0725	.03040	.93239	.0701	2̄.84545	1.0025	.00106	.06989
0.08	1.0833	.03474	.92312	.0801	2̄.90355	1.0032	.00139	.07983
0.09	1.0942	.03909	.91393	.0901	2̄.95483	1.0041	.00176	.08976
0.10	1.1052	.04343	.90484	.1002	1̄.00072	1.0050	.00217	.09967
0.11	1.1163	.04777	.89583	.1102	1̄.04227	1.0061	.00262	.10956
0.12	1.1275	.05212	.88692	.1203	1̄.08022	1.0072	.00312	.11943
0.13	1.1388	.05646	.87809	.1304	1̄.11517	1.0085	.00366	.12927
0.14	1.1503	.06080	.86936	.1405	1̄.14755	1.0098	.00424	.13909
0.15	1.1618	.06514	.86071	.1506	1̄.17772	1.0113	.00487	.14889
0.16	1.1735	.06949	.85214	.1607	1̄.20597	1.0128	.00554	.15865
0.17	1.1853	.07383	.84366	.1708	1̄.23254	1.0145	.00625	.16838
0.18	1.1972	.07817	.83527	.1810	1̄.25762	1.0162	.00700	.17808
0.19	1.2092	.08252	.82696	.1911	1̄.28136	1.0181	.00779	.18775
0.20	1.2214	.08686	.81873	.2013	1̄.30392	1.0201	.00863	.19738
0.21	1.2337	.09120	.81058	.2115	1̄.32541	1.0221	.00951	.20697
0.22	1.2461	.09554	.80252	.2218	1̄.34592	1.0243	.01043	.21652
0.23	1.2586	.09989	.79453	.2320	1̄.36555	1.0266	.01139	.22603
0.24	1.2712	.10423	.78663	.2423	1̄.38437	1.0289	.01239	.23550
0.25	1.2840	.10857	.77880	.2526	1̄.40245	1.0314	.01343	.24492
0.26	1.2969	.11292	.77105	.2629	1̄.41986	1.0340	.01452	.25430
0.27	1.3100	.11726	.76338	.2733	1̄.43663	1.0367	.01564	.26362
0.28	1.3231	.12160	.75578	.2837	1̄.45282	1.0395	.01681	.27291
0.29	1.3364	.12595	.74826	.2941	1̄.46847	1.0423	.01801	.28213
0.30	1.3499	.13029	.74082	.3045	1̄.48362	1.0453	.01926	.29131
0.31	1.3634	.13463	.73345	.3150	1̄.49830	1.0484	.02054	.30044
0.32	1.3771	.13897	.72615	.3255	1̄.51254	1.0516	.02187	.30951
0.33	1.3910	.14332	.71892	.3360	1̄.52637	1.0549	.02323	.31852
0.34	1.4049	.14766	.71177	.3466	1̄.53981	1.0584	.02463	.32748
0.35	1.4191	.15200	.70469	.3572	1̄.55290	1.0619	.02607	.33638
0.36	1.4333	.15635	.69768	.3678	1̄.56564	1.0655	.02755	.34521
0.37	1.4477	.16069	.69073	.3785	1̄.57807	1.0692	.02907	.35399
0.38	1.4623	.16503	.68386	.3892	1̄.59019	1.0731	.03063	.36271
0.39	1.4770	.16937	.67706	.4000	1̄.60202	1.0770	.03222	.37136
0.40	1.4918	.17372	.67032	.4108	1̄.61358	1.0811	.03385	.37995
0.41	1.5063	.17806	.66365	.4216	1̄.62488	1.0852	.03552	.38847
0.42	1.5220	.18240	.65705	.4325	1̄.63594	1.0895	.03723	.39693
0.43	1.5373	.18675	.65051	.4434	1̄.64677	1.0939	.03897	.40532
0.44	1.5527	.19109	.64404	.4543	1̄.65738	1.0984	.04075	.41364
0.45	1.5683	.19543	.63763	.4653	1̄.66777	1.1030	.04256	.42190
0.46	1.5841	.19978	.63128	.4764	1̄.67797	1.1077	.04441	.43008
0.47	1.6000	.20412	.62500	.4875	1̄.68797	1.1125	.04630	.43820
0.48	1.6161	.20846	.61878	.4986	1̄.69779	1.1174	.04822	.44624
0.49	1.6323	.21280	.61263	.5098	1̄.70744	1.1225	.05018	.45422
0.50	1.6487	.21715	.60653	.5211	1̄.71692	1.1276	.05217	.46212
0.51	1.6653	.22149	.60050	.5324	1̄.72624	1.1329	.05419	.46995
0.52	1.6820	.22583	.59452	.5438	1̄.73540	1.1383	.05625	.47770
0.53	1.6989	.23018	.58860	.5552	1̄.74442	1.1438	.05834	.48538
0.54	1.7160	.23452	.58275	.5666	1̄.75330	1.1494	.06046	.49299
0.55	1.7333	.23886	.57695	.5782	1̄.76204	1.1551	.06262	.50052
0.56	1.7507	.24320	.57121	.5897	1̄.77065	1.1609	.06481	.50798
0.57	1.7683	.24755	.56553	.6014	1̄.77914	1.1669	.06703	.51536
0.58	1.7860	.25189	.55990	.6131	1̄.78751	1.1730	.06929	.52267
0.59	1.8040	.25623	.55433	.6248	1̄.79576	1.1792	.07157	.52990
0.60	1.8221	.26058	.54881	.6367	1̄.80390	1.1855	.07389	.53705
0.61	1.8404	.26492	.54335	.6485	1̄.81194	1.1919	.07624	.54413
0.62	1.8589	.26926	.53794	.6605	1̄.81987	1.1984	.07861	.55113
0.63	1.8776	.27361	.53259	.6725	1̄.82770	1.2051	.08102	.55805
0.64	1.8965	.27795	.52729	.6846	1̄.83543	1.2119	.08346	.56490
0.65	1.9155	.28229	.52205	.6967	1̄.84308	1.2188	.08593	.57167
0.66	1.9348	.28664	.51685	.7090	1̄.85063	1.2258	.08843	.57836
0.67	1.9542	.29098	.51171	.7213	1̄.85809	1.2330	.09095	.58498
0.68	1.9739	.29532	.50662	.7336	1̄.86548	1.2402	.09351	.59152
0.69	1.9937	.29966	.50158	.7461	1̄.87278	1.2476	.09609	.59798
0.70	2.0138	.30401	.49659	.7586	1̄.88000	1.2552	.09870	.60437
0.71	2.0340	.30835	.49164	.7712	1̄.88715	1.2628	.10134	.61068
0.72	2.0544	.31269	.48675	.7838	1̄.89423	1.2706	.10401	.61691

EXPONENTIAL AND HYPERBOLIC FUNCTIONS AND THEIR COMMON LOGARITHMS

(continued)

x	e^x		e^{-x} (value)	$\sinh x$		$\cosh x$		$\tanh x$ (value)
	Value	$\log^{10}$		Value	$\log^{10}$	Value	$\log^{10}$	
0.73	2.0751	.31703	.48191	.7966	̄.90123	1.2785	.10670	.62307
0.74	2.0959	.32138	.47711	.8094	̄.90817	1.2865	.10942	.62915
0.75	2.1170	.32572	.47237	.8223	̄.91504	1.2947	.11216	.63515
0.76	2.1383	.33006	.46767	.8353	̄.92185	1.3030	.11493	.64108
0.77	2.1598	.33441	.46301	.8484	̄.92859	1.3114	.11773	.64693
0.78	2.1815	.33875	.45841	.8615	̄.93527	1.3199	.12055	.65721
0.79	2.2034	.34309	.45384	.8748	̄.94190	1.3286	.12340	.65841
0.80	2.2255	.34744	.44933	.8881	̄.94846	1.3374	.12627	.66404
0.81	2.2479	.35178	.44486	.9015	̄.95498	1.3464	.12917	.66959
0.82	2.2705	.35612	.44043	.9150	̄.96144	1.3555	.13209	.67507
0.83	2.2933	.36046	.43605	.9286	̄.96784	1.3647	.13503	.68048
0.84	2.3164	.36481	.43171	.9423	̄.97420	1.3740	.13800	.68581
0.85	2.3396	.36915	.42741	.9561	̄.98051	1.3835	.14099	.69107
0.86	2.3632	.37349	.42316	.9700	̄.98677	1.3932	.14400	.69626
0.87	2.3869	.37784	.41895	.9840	̄.99299	1.4029	.14704	.70137
0.88	2.4100	.38218	.41478	.9981	̄.99916	1.4128	.15009	.70642
0.89	2.4351	.38652	.41066	1.0122	0.00528	1.4229	.15317	.71139
0.90	2.4596	.39087	.40657	1.0265	.01137	1.4331	.15627	.71630
0.91	2.4843	.39521	.40242	1.0409	.01741	1.4434	.15939	.72113
0.92	2.5093	.39955	.39852	1.0554	.02341	1.4539	.16254	.72590
0.93	2.5345	.40389	.39455	1.0700	.02937	1.4645	.16570	.73059
0.94	2.5600	.40824	.39063	1.0847	.03530	1.4753	.16888	.73522
0.95	2.5857	.41258	.38674	1.0995	.04119	1.4862	.17208	.73978
0.96	2.6117	.41692	.38289	1.1144	.04704	1.4973	.17531	.74428
0.97	2.6379	.42127	.37908	1.1294	.05286	1.5085	.17855	.74870
0.98	2.6645	.42561	.37531	1.1446	.05864	1.5199	.18181	.75307
0.99	2.6912	.42995	.37158	1.1598	.06439	1.5314	.18509	.75736
1.00	2.7183	.43429	.36788	1.1752	.07011	1.5431	.18839	.76159
1.01	2.7456	.43864	.36422	1.1907	.07580	1.5549	.19171	.76576
1.02	2.7732	.44298	.36060	1.2063	.06146	1.5669	.19504	.76987
1.03	2.8011	.44732	.35701	1.2220	.08708	1.5790	.19839	.77391
1.04	2.8292	.45167	.35345	1.2379	.09268	1.5913	.20176	.77789
1.05	2.8577	.45601	.34994	1.2539	.09825	1.6038	.20515	.78181
1.06	2.8864	.46035	.34646	1.2700	.10379	1.6164	.20855	.78566
1.07	2.9154	.46470	.34301	1.2862	.10930	1.6292	.21197	.78946
1.08	2.9447	.46904	.33960	1.3025	.11479	1.6421	.21541	.79320
1.09	2.9743	.47338	.33622	1.3190	.12025	1.6552	.21886	.79688
1.10	3.0042	.47772	.33287	1.3356	.12569	1.6685	.22233	.80050
1.11	3.0344	.48207	.32956	1.3524	.13111	1.6820	.22582	.80406
1.12	3.0659	.48641	.32628	1.3693	.13649	1.6956	.22931	.80757
1.13	3.0977	.49075	.32303	1.3863	.14186	1.7083	.23283	.81102
1.14	3.1298	.49510	.31982	1.4035	.14720	1.7233	.23636	.81441
1.15	3.1622	.49944	.31664	1.4208	.15253	1.7374	.23990	.81775
1.16	3.1899	.50378	.31349	1.4382	.15783	1.7517	.24346	.82104
1.17	3.2220	.50812	.31037	1.4558	.16311	1.7662	.24703	.82427
1.18	3.2544	.51247	.30728	1.4735	.16836	1.7808	.25062	.82745
1.19	3.2871	.51681	.30422	1.4914	.17360	1.7957	.25422	.83058
1.20	3.3201	.52115	.30119	1.5095	.17882	1.8107	.25784	.83365
1.21	3.3535	.52550	.29820	1.5276	.18402	1.8258	.26146	.83668
1.22	3.3872	.52984	.29523	1.5460	.18920	1.8412	.26510	.83965
1.23	3.4212	.53418	.29229	1.5645	.19437	1.8568	.26876	.84258
1.24	3.4556	.53853	.28938	1.5831	.19951	1.8725	.27242	.84546
1.25	3.4903	.54287	.28650	1.6019	.20464	1.8884	.27610	.84828
1.26	3.5254	.54721	.28365	1.6209	.20975	1.9045	.27979	.85106
1.27	3.5609	.55155	.28083	1.6400	.21485	1.9208	.28349	.85380
1.28	3.5966	.55590	.27804	1.6593	.21993	1.9373	.28721	.85648
1.29	3.6328	.56024	.27527	1.6788	.22499	1.9540	.29093	.85913
1.30	3.6693	.56458	.27253	1.6984	.23004	1.9709	.29467	.86172
1.31	3.7062	.56893	.26982	1.7182	.23507	1.9880	.29842	.86428
1.32	3.7434	.57327	.26714	1.7381	.24009	2.0053	.30217	.86678
1.33	3.7810	.57761	.26448	1.7583	.24509	2.0228	.30594	.86925
1.34	3.8190	.58195	.26185	1.7786	.25008	2.0404	.30972	.87167
1.35	3.8574	.58630	.25924	1.7991	.25505	2.0583	.31352	.87405
1.36	3.8962	.59064	.25666	1.8198	.26002	2.0764	.31732	.87639
1.37	3.9354	.59498	.25411	1.8406	.26496	2.0947	.32113	.87869
1.38	3.9749	.59933	.25158	1.8617	.26990	2.1132	.32495	.88095
1.39	4.0149	.60367	.24908	1.8829	.27482	2.1320	.32878	.88317
1.40	4.0552	.60801	.24660	1.9043	.27974	2.1509	.33262	.88535
1.41	4.0960	.61236	.24414	1.9259	.28464	2.1700	.33647	.88749
1.42	4.1371	.61670	.24171	1.9477	.28952	2.1894	.34033	.88960
1.43	4.1787	.62104	.23931	1.9697	.29440	2.2090	.34420	.89167

EXPONENTIAL AND HYPERBOLIC FUNCTIONS AND THEIR COMMON LOGARITHMS

(continued)

x	e^x		e^{-x} (value)	$\sinh x$		$\cosh x$		$\tanh x$ (value)
	Value	$\log^{10}$		Value	$\log^{10}$	Value	$\log^{10}$	
1.44	4.2207	.62538	.23693	1.9919	.29926	2.2288	.34807	.89370
1.45	4.2631	.62973	.23457	2.0143	.30412	2.2488	.35196	.89569
1.46	4.3060	.63407	.23224	2.0369	.30896	2.2691	.35585	.89765
1.47	4.3492	.63841	.22993	2.0597	.31379	2.2896	.35976	.89958
1.48	4.3929	.64276	.22764	2.0827	.31862	2.3103	.36367	.90147
1.49	4.4371	.64710	.22537	2.1059	.32343	2.3312	.36759	.90332
1.50	4.4817	.65144	.22313	2.1293	.32823	2.3524	.37151	.90515
1.51	4.5267	.65578	.22091	2.1529	.33303	2.3738	.37545	.90694
1.52	4.5722	.66013	.21871	2.1768	.33781	2.3955	.37939	.90870
1.53	4.6182	.66447	.21654	2.2008	.34258	2.4174	.38334	.91042
1.54	4.6646	.66881	.21438	2.2251	.34735	2.4395	.38730	.91212
1.55	4.7115	.67316	.21225	2.2496	.35211	2.4619	.39126	.91379
1.56	4.7588	.67750	.21014	2.2743	.35686	2.4845	.39524	.91542
1.57	4.8066	.68184	.20805	2.2993	.36160	2.5073	.39921	.91703
1.58	4.8550	.68619	.20598	2.3245	.36633	2.5305	.40320	.91860
1.59	4.9037	.69053	.20393	2.3499	.37105	2.5538	.40719	.92015
1.60	4.9530	.69487	.20190	2.3756	.37577	2.5775	.41119	.92167
1.61	5.0028	.69921	.19989	2.4015	.38048	2.6013	.41520	.92316
1.62	5.0531	.70356	.19790	2.4276	.38518	2.6255	.41921	.92462
1.63	5.1039	.70790	.19593	2.4540	.38987	2.6499	.42323	.92606
1.64	5.1552	.71224	.19398	2.4806	.39456	2.6746	.42725	.92747
1.65	5.2070	.71659	.19205	2.5075	.39923	2.6995	.43129	.92886
1.66	5.2593	.72093	.19014	2.5346	.40391	2.7247	.43532	.93022
1.67	5.3122	.72527	.18825	2.5620	.40857	2.7502	.43937	.93155
1.68	5.3656	.72961	.18637	2.5896	.41323	2.7760	.44341	.93286
1.69	5.4195	.73396	.18452	2.6175	.41788	2.8020	.44747	.93415
1.70	5.4739	.73830	.18268	2.6456	.42253	2.8283	.45153	.93541
1.71	5.5290	.74264	.18087	2.6740	.42717	2.8549	.45559	.93665
1.72	5.5845	.74699	.17907	2.7027	.43180	2.8818	.45966	.93786
1.73	5.6407	.75133	.17728	2.7317	.43643	2.9090	.46374	.93906
1.74	5.6973	.75567	.17552	2.7609	.44105	2.9364	.46782	.94023
1.75	5.7546	.76002	.17377	2.7904	.44567	2.9642	.47191	.94138
1.76	5.8124	.76436	.17204	2.8202	.45028	2.9922	.47600	.94250
1.77	5.8709	.76870	.17033	2.8503	.45488	3.0206	.48009	.94361
1.78	5.9299	.77304	.16864	2.8806	.45948	3.0492	.48419	.94470
1.79	5.9895	.77739	.16696	2.9112	.46408	3.0782	.48830	.94576
1.80	6.0496	.78173	.16530	2.9422	.46867	3.1075	.49241	.94681
1.81	6.1104	.78607	.16365	2.9734	.47325	3.1371	.49652	.94783
1.82	6.1719	.79042	.16203	3.0049	.47783	3.1669	.50064	.94884
1.83	6.2339	.79476	.16041	3.0367	.48241	3.1972	.50476	.94983
1.84	6.2965	.79910	.15882	3.0689	.48698	3.2277	.50889	.95080
1.85	6.3598	.80344	.15724	3.1013	.49154	3.2585	.51302	.95175
1.86	6.4237	.80779	.15567	3.1340	.49610	3.2897	.51716	.95268
1.87	6.4383	.81213	.15412	3.1671	.50066	3.3212	.52130	.95359
1.88	6.5535	.81647	.15259	3.2005	.50521	3.3530	.52544	.95449
1.89	6.6194	.82082	.15107	3.2341	.50976	3.3852	.52959	.95537
1.90	6.6859	.82516	.14957	3.2682	.51430	3.4177	.53374	.95624
1.91	6.7531	.82950	.14808	3.3025	.51884	3.4506	.53789	.95709
1.92	6.8210	.83385	.14661	3.3372	.52338	3.4838	.54205	.95792
1.93	6.8895	.83819	.14515	3.3722	.52791	3.5173	.54621	.95873
1.94	6.9588	.84253	.14370	3.4075	.53244	3.5512	.55038	.95953
1.95	7.0287	.84687	.14227	3.4432	.53696	3.5855	.55455	.96032
1.96	7.0993	.85122	.14086	3.4792	.54148	3.6201	.55872	.96109
1.97	7.1707	.85556	.13946	3.5156	.54600	3.6551	.56290	.96185
1.98	7.2427	.85990	.13807	3.5923	.55051	3.6904	.56707	.96259
1.99	7.3155	.86425	.13670	3.5894	.55502	3.7261	.57126	.96331
2.00	7.3891	.86859	.13534	3.6269	.55953	3.7622	.57544	.96403
2.01	7.4633	.87293	.13399	3.6647	.56403	3.7987	.57963	.96473
2.02	7.5383	.87727	.13266	3.7028	.56853	3.8335	.58382	.96541
2.03	7.6141	.88162	.13134	3.7414	.57303	3.8727	.58802	.96609
2.04	7.6906	.88596	.13003	3.7803	.57753	3.9103	.59221	.96675
2.05	7.7679	.89030	.12873	3.8196	.58202	3.9483	.59641	.96740
2.06	7.8460	.89465	.12745	3.8593	.58650	3.9867	.60061	.96803
2.07	7.9248	.89899	.12619	3.8993	.59099	4.0255	.60482	.96865
2.08	8.0045	.90333	.12493	3.9398	.59547	4.0647	.60903	.96926
2.09	8.0849	.90768	.12369	3.9806	.59995	4.1043	.61324	.96986
2.10	8.1662	.91202	.12246	4.0219	.60443	4.1443	.61745	.97045
2.11	8.2482	.91636	.12124	4.0635	.60890	4.1847	.62167	.97103
2.12	8.3311	.92070	.12003	4.1056	.61337	4.2256	.62589	.97159
2.13	8.4149	.92505	.11884	4.1480	.61784	4.2669	.63011	.97215
2.14	8.4994	.92939	.11765	4.1909	.62231	4.3085	.63433	.97269
2.15	8.5849	.93373	.11648	4.2342	.62677	4.3507	.63856	.97323

EXPONENTIAL AND HYPERBOLIC FUNCTIONS AND THEIR COMMON LOGARITHMS

(continued)

x	e ^x		e ^{-x} (value)	sinh x		cosh x		tanh x (value)
	Value	log ¹⁰		Value	log ¹⁰	Value	log ¹⁰	
2.16	8.6711	.93808	.11533	4.2779	.63123	4.3932	.64278	.97375
2.17	8.7583	.94242	.11418	4.3221	.63569	4.4362	.64701	.97426
2.18	8.8463	.94676	.11304	4.3666	.64015	4.4797	.65125	.97477
2.19	8.9352	.95110	.11192	4.4116	.64460	4.5236	.65548	.97526
2.20	9.0250	.95545	.11080	4.4571	.64905	4.5679	.65972	.97574
2.21	9.1157	.95979	.10970	4.5030	.65350	4.6127	.66396	.97622
2.22	9.2073	.96413	.10861	4.5494	.65795	4.6580	.66820	.97668
2.23	9.2999	.96848	.10753	4.5962	.66240	4.7037	.67244	.97714
2.24	9.3933	.97282	.10646	4.6434	.66684	4.7499	.67668	.97759
2.25	9.4877	.97716	.10540	4.6912	.67128	4.7966	.68093	.97803
2.26	9.5831	.98151	.10435	4.7394	.67572	4.8437	.68518	.97846
2.27	9.6794	.98585	.10331	4.7880	.68016	4.8914	.68943	.97888
2.28	9.7767	.99019	.10228	4.8372	.68459	4.9395	.69368	.97929
2.29	9.8749	.99453	.10127	4.8868	.68903	4.9881	.69794	.97970
2.30	9.9742	.99888	.10026	4.9370	.69346	5.0372	.70219	.98010
2.31	10.074	1.00322	.09926	4.9876	.69789	5.0868	.70645	.98049
2.32	10.176	1.00756	.09827	5.0387	.70232	5.1370	.71071	.98087
2.33	10.278	1.01191	.09730	5.0903	.70675	5.1876	.71497	.98124
2.34	10.381	1.01625	.09633	5.1425	.71117	5.2388	.71923	.98161
2.35	10.486	1.02059	.09537	5.1951	.71559	5.2905	.72349	.98197
2.36	10.591	1.02493	.09442	5.2483	.72002	5.3427	.72776	.98233
2.37	10.697	1.02928	.09348	5.3020	.72444	5.3954	.73203	.98267
2.38	10.805	1.03362	.09255	5.3562	.72885	5.4487	.73630	.98301
2.39	10.913	1.03796	.09163	5.4109	.73327	5.5026	.74056	.98335
2.40	11.023	1.04231	.09072	5.4662	.73769	5.5569	.74484	.98367
2.41	11.134	1.04665	.08982	5.5221	.74210	5.6119	.74911	.98400
2.42	11.246	1.05099	.08892	5.5785	.74652	5.6674	.75338	.98431
2.43	11.359	1.05534	.08804	5.6354	.75093	5.7235	.75766	.98462
2.44	11.473	1.05968	.08716	5.6929	.75534	5.7801	.76194	.98492
2.45	11.588	1.06402	.08629	5.7510	.75975	5.8373	.76621	.98522
2.46	11.705	1.06836	.08543	5.8097	.76415	5.8951	.77049	.98551
2.47	11.822	1.07271	.08458	5.8689	.76856	5.9535	.77477	.98579
2.48	11.941	1.07705	.08374	5.9288	.77296	6.0125	.77906	.98607
2.49	12.061	1.08139	.08291	5.9892	.77737	6.0721	.78334	.98635
2.50	12.182	1.08574	.08208	6.0502	.78177	6.1323	.78762	.98661
2.51	12.305	1.09008	.08127	6.1118	.78617	6.1931	.79191	.98688
2.52	12.429	1.09442	.08046	6.1741	.79057	6.2545	.79619	.98714
2.53	12.554	1.09877	.07966	6.2369	.79497	6.3166	.80048	.98739
2.54	12.680	1.10311	.07887	6.3004	.79937	6.3793	.80477	.98764
2.55	12.807	1.10745	.07808	6.3645	.80377	6.4426	.80906	.98788
2.56	12.936	1.11179	.07730	6.4293	.80816	6.5066	.81335	.98812
2.57	13.066	1.11614	.07654	6.4946	.81256	6.5712	.81764	.98835
2.58	13.197	1.12048	.07577	6.5607	.81695	6.6365	.82194	.98858
2.59	13.330	1.12482	.07502	6.6274	.82134	6.7024	.82623	.98881
2.60	13.464	1.12917	.07427	6.6947	.82573	6.7690	.83052	.98903
2.61	13.599	1.13351	.07353	6.7628	.83012	6.8363	.83482	.98924
2.62	13.736	1.13785	.07280	6.8315	.83451	6.9043	.83912	.98946
2.63	13.874	1.14219	.07208	6.9008	.83890	6.9729	.84341	.98966
2.64	14.013	1.14654	.07136	6.9709	.84329	7.0423	.84771	.98987
2.65	14.154	1.15008	.07065	7.0417	.84768	7.1123	.85201	.99007
2.66	14.296	1.15522	.06995	7.1132	.85206	7.1831	.85631	.99026
2.67	14.440	1.15957	.06925	7.1854	.85645	7.2546	.86061	.99045
2.68	14.585	1.16391	.06856	7.2583	.86083	7.3268	.86492	.99064
2.69	14.732	1.16825	.06788	7.3319	.86522	7.3998	.86922	.99083
2.70	14.880	1.17260	.06721	7.4063	.86960	7.4735	.87352	.99101
2.71	15.029	1.17694	.06654	7.4814	.87398	7.5479	.87783	.99118
2.72	15.180	1.18128	.06587	7.5572	.87836	7.6231	.88213	.99136
2.73	15.333	1.18562	.06522	7.6338	.88274	7.6991	.88644	.99153
2.74	15.487	1.18997	.06457	7.7112	.88712	7.7758	.89074	.99170
2.75	15.643	1.19431	.06393	7.7894	.89150	7.8533	.89505	.99186
2.76	15.800	1.19865	.06329	7.8683	.89588	7.9316	.89936	.99202
2.77	15.959	1.20300	.06266	7.9480	.90026	8.0106	.90367	.99218
2.78	16.119	1.20734	.06204	8.0285	.90463	8.0905	.90798	.99233
2.79	16.281	1.21168	.06142	8.1098	.90901	8.1712	.91229	.99248
2.80	16.445	1.21602	.06081	8.1919	.91339	8.2527	.91660	.99263
2.81	16.610	1.22037	.06020	8.2749	.91776	8.3351	.92091	.99278
2.82	16.777	1.22471	.05961	8.3586	.92213	8.4182	.92522	.99292
2.83	16.945	1.22905	.05901	8.4432	.92651	8.5022	.92953	.99306
2.84	17.116	1.23340	.05843	8.5287	.93088	8.5871	.93385	.99320
2.85	17.288	1.23774	.05784	8.6150	.93525	8.6728	.93816	.99333
2.86	17.462	1.24208	.05727	8.7021	.93963	8.7594	.94247	.99346

EXPONENTIAL AND HYPERBOLIC FUNCTIONS AND THEIR COMMON LOGARITHMS

(continued)

x	e ^x		e ^{-x} (value)	sinh x		cosh x		tanh x (value)
	Value	log ¹⁰		Value	log ¹⁰	Value	log ¹⁰	
2.87	17.637	1.24643	.05670	8.7902	.94400	8.8469	.94679	.99359
2.88	17.814	1.25077	.05613	8.8791	.94837	8.9352	.95110	.99372
2.89	17.993	1.25511	.05558	8.9689	.95274	9.0244	.95542	.99384
2.90	18.174	1.25945	.05502	9.0596	.95711	9.1146	.95974	.99396
2.91	18.357	1.26380	.05448	9.1512	.96148	9.2056	.96405	.99408
2.92	18.541	1.26814	.05393	9.2437	.96584	9.2976	.96837	.99420
2.93	18.728	1.27248	.05340	9.3371	.97021	9.3905	.97269	.99431
2.94	18.916	1.27683	.05287	9.4315	.97458	9.4844	.97701	.99443
2.95	19.106	1.28117	.05234	9.5268	.97895	9.5791	.98133	.99454
2.96	19.298	1.28551	.05182	9.6231	.98331	9.6749	.98565	.99464
2.97	19.492	1.28985	.05130	9.7203	.98768	9.7716	.98997	.99475
2.98	19.688	1.29420	.05079	9.8185	.99205	9.8693	.99429	.99485
2.99	19.886	1.29854	.05029	9.9177	.99641	9.9680	.99861	.99496
3.00	20.086	1.30288	.04979	10.018	1.00078	10.068	1.00293	0.99505
3.05	21.115	1.32460	.04736	10.534	1.02259	10.581	1.02454	0.99552
3.10	22.198	1.34631	.04505	11.076	1.04440	11.122	1.04616	0.99595
3.15	23.336	1.36803	.04285	11.647	1.06620	11.690	1.06779	0.99633
3.20	24.533	1.38974	.04076	12.246	1.08799	12.287	1.08943	0.99668
3.25	25.790	1.41146	.03877	12.876	1.10977	12.915	1.11108	0.99700
3.30	27.113	1.43317	.03688	13.538	1.13155	13.575	1.13273	0.99728
3.35	28.503	1.45489	.03508	14.234	1.15332	14.269	1.15439	0.99754
3.40	29.964	1.47660	.03337	14.965	1.17509	14.999	1.17605	0.99777
3.45	31.500	1.49832	.03175	15.734	1.19685	15.766	1.19772	0.99799
3.50	33.115	1.52003	.03020	16.543	1.21860	16.573	1.21940	0.99818
3.55	34.813	1.54175	.02872	17.392	1.24036	17.421	1.24107	0.99835
3.60	36.598	1.56346	.02732	18.286	1.26211	18.313	1.26275	0.99851
3.65	38.475	1.58517	.02599	19.224	1.28385	19.250	1.28444	0.99865
3.70	40.447	1.60689	.02472	20.211	1.30559	20.236	1.30612	0.99878
3.75	42.521	1.62860	.02352	21.249	1.32733	21.272	1.32781	0.99889
3.80	44.701	1.65032	.02237	22.339	1.34907	22.362	1.34951	0.99900
3.85	46.993	1.67203	.02128	23.486	1.37081	23.507	1.37120	0.99909
3.90	49.402	1.69375	.02024	24.691	1.39254	24.711	1.39290	0.99918
3.95	51.935	1.71546	.01925	25.958	1.41427	25.977	1.41459	0.99926
4.00	54.598	1.73718	.01832	27.290	1.43600	27.308	1.43629	0.99933
4.10	60.340	1.78061	.01657	30.162	1.47946	30.178	1.47970	0.99945
4.20	66.686	1.82404	.01500	33.336	1.52291	33.351	1.52310	0.99955
4.30	73.700	1.86747	.01357	36.843	1.56636	36.857	1.56652	0.99963
4.40	81.451	1.91090	.01227	40.719	1.60980	40.732	1.60993	0.99970
4.50	90.017	1.95433	.01111	45.003	1.65324	45.014	1.65335	0.99975
4.60	99.484	1.99775	.01005	49.737	1.69668	49.747	1.69677	0.99980
4.70	109.95	2.04118	.00910	54.969	1.74012	54.978	1.74019	0.99983
4.80	121.51	2.08461	.00823	60.751	1.78355	60.759	1.78361	0.99986
4.90	134.29	2.12804	.00745	67.141	1.82699	67.149	1.82704	0.99989
5.00	148.41	2.17147	.00674	74.203	1.87042	74.210	1.87046	0.99991
5.10	164.02	2.21490	.00610	82.008	1.91389	82.014	1.91389	0.99993
5.20	181.27	2.25833	.00552	90.633	1.95729	90.639	1.95731	0.99994
5.30	200.34	2.30176	.00499	100.17	2.00074	100.17	2.00074	0.99995
5.40	221.41	2.34519	.00452	110.70	2.04415	110.71	2.04417	0.99996
5.50	244.69	2.38862	.00409	122.34	2.08758	122.35	2.08760	0.99997
5.60	270.43	2.43205	.00370	135.21	2.13101	135.22	2.13103	0.99997
5.70	298.87	2.47548	.00335	149.43	2.17444	149.44	2.17445	0.99998
5.80	330.30	2.51891	.00303	165.15	2.21787	165.15	2.21788	0.99998
5.90	365.04	2.56234	.00274	182.52	2.26130	182.52	2.26131	0.99998
6.00	403.43	2.60577	.00248	201.71	2.30473	201.72	2.30474	0.99999
6.25	518.01	2.71434	.00193	259.01	2.41331	259.01	2.41331	0.99999
6.50	665.14	2.82291	.00150	332.57	2.52188	332.57	2.52189	1.00000
6.75	854.06	2.93149	.00117	427.03	2.63046	427.03	2.63046	1.00000
7.00	1096.6	3.04006	.00091	548.32	2.73904	548.32	2.73903	1.00000
7.50	1808.0	3.25721	.00055	904.02	2.95618	904.02	2.95618	1.00000
8.00	2981.0	3.47436	.00034	1490.5	3.17333	1490.5	3.17333	1.00000
8.50	4914.8	3.69150	.00020	2457.4	3.39047	2457.4	3.39047	1.00000
9.00	8103.1	3.90865	.00012	4051.5	3.60762	4051.5	3.60762	1.00000
9.50	13360.	4.12580	.00007	6679.9	3.82477	6679.9	3.82477	1.00000
10.00	22026.	4.34294	.00005	11013.	4.04191	11013.	4.04191	1.00000

RELATION OF ANGULAR FUNCTIONS IN TERMS OF ONE ANOTHER

TRIGONOMETRIC FUNCTIONS

Function	$\sin \alpha$	$\cos \alpha$	$\tan \alpha$	$\cot \alpha$	$\sec \alpha$	$\csc \alpha$
$\sin \alpha$	$\sin \alpha$	$\pm\sqrt{1 - \cos^2 \alpha}$	$\frac{\tan \alpha}{\pm\sqrt{1 + \tan^2 \alpha}}$	$\frac{1}{\pm\sqrt{1 + \cot^2 \alpha}}$	$\frac{\pm\sqrt{\sec^2 \alpha - 1}}{\sec \alpha}$	$\frac{1}{\csc \alpha}$
$\cos \alpha$	$\pm\sqrt{1 - \sin^2 \alpha}$	$\cos \alpha$	$\frac{1}{\pm\sqrt{1 + \tan^2 \alpha}}$	$\frac{\cot \alpha}{\pm\sqrt{1 + \cot^2 \alpha}}$	$\frac{1}{\sec \alpha}$	$\frac{\pm\sqrt{\csc^2 \alpha - 1}}{\csc \alpha}$
$\tan \alpha$	$\frac{\sin \alpha}{\pm\sqrt{1 - \sin^2 \alpha}}$	$\frac{\pm\sqrt{1 - \cos^2 \alpha}}{\cos \alpha}$	$\tan \alpha$	$\frac{1}{\cot \alpha}$	$\pm\sqrt{\sec^2 \alpha - 1}$	$\frac{1}{\pm\sqrt{\csc^2 \alpha - 1}}$
$\cot \alpha$	$\frac{\pm\sqrt{1 - \sin^2 \alpha}}{\sin \alpha}$	$\frac{\cos \alpha}{\pm\sqrt{1 - \cos^2 \alpha}}$	$\frac{1}{\tan \alpha}$	$\cot \alpha$	$\frac{1}{\pm\sqrt{\sec^2 \alpha - 1}}$	$\pm\sqrt{\csc^2 \alpha - 1}$
$\sec \alpha$	$\frac{1}{\pm\sqrt{1 - \sin^2 \alpha}}$	$\frac{1}{\cos \alpha}$	$\pm\sqrt{1 + \tan^2 \alpha}$	$\frac{\pm\sqrt{1 + \cot^2 \alpha}}{\cot \alpha}$	$\sec \alpha$	$\frac{\csc \alpha}{\pm\sqrt{\csc^2 \alpha - 1}}$
$\csc \alpha$	$\frac{1}{\sin \alpha}$	$\frac{1}{\pm\sqrt{1 - \cos^2 \alpha}}$	$\frac{\pm\sqrt{1 + \tan^2 \alpha}}{\tan \alpha}$	$\pm\sqrt{1 + \cot^2 \alpha}$	$\frac{\sec \alpha}{\pm\sqrt{\sec^2 \alpha - 1}}$	$\csc \alpha$

Note: The choice of sign depends upon the quadrant in which the angle terminates.

HYPERBOLIC FUNCTIONS

Function	$\sinh x$	$\cosh x$	$\tanh x$
$\sinh x =$	$\sinh x$	$\pm\sqrt{\cosh^2 x - 1}$	$\frac{\tanh x}{\sqrt{1 - \tanh^2 x}}$
$\cosh x =$	$\sqrt{1 + \sinh^2 x}$	$\cosh x$	$\frac{1}{\sqrt{1 - \tanh^2 x}}$
$\tanh x =$	$\frac{\sinh x}{\sqrt{1 + \sinh^2 x}}$	$\pm\frac{\sqrt{\cosh^2 x - 1}}{\cosh x}$	$\tanh x$
$\operatorname{cosech} x =$	$\frac{1}{\sinh x}$	$\pm\frac{1}{\sqrt{\cosh^2 x - 1}}$	$\frac{\sqrt{1 - \tanh^2 x}}{\tanh x}$
$\operatorname{sech} x =$	$\frac{1}{\sqrt{1 + \sinh^2 x}}$	$\frac{1}{\cosh x}$	$\sqrt{1 - \tanh^2 x}$
$\operatorname{coth} x =$	$\frac{\sqrt{1 + \sinh^2 x}}{\sinh x}$	$\frac{\pm \cosh x}{\sqrt{\cosh^2 x - 1}}$	$\frac{1}{\tanh x}$
Function	$\operatorname{cosech} x$	$\operatorname{sech} x$	$\operatorname{coth} x$
$\sinh x =$	$\frac{1}{\operatorname{cosech} x}$	$\pm\frac{\sqrt{1 - \operatorname{sech}^2 x}}{\operatorname{sech} x}$	$\frac{\pm 1}{\sqrt{\operatorname{coth}^2 x - 1}}$
$\cosh x =$	$\pm\frac{\sqrt{\operatorname{cosech}^2 x + 1}}{\operatorname{cosech} x}$	$\frac{1}{\operatorname{sech} x}$	$\pm\frac{\operatorname{coth} x}{\sqrt{\operatorname{coth}^2 x - 1}}$
$\tanh x =$	$\frac{1}{\sqrt{\operatorname{cosech}^2 x + 1}}$	$\pm\sqrt{1 - \operatorname{sech}^2 x}$	$\frac{1}{\operatorname{coth} x}$
$\operatorname{cosech} x =$	$\operatorname{cosech} x$	$\pm\frac{\operatorname{sech} x}{\sqrt{1 - \operatorname{sech}^2 x}}$	$\pm\frac{\sqrt{\operatorname{coth}^2 x - 1}}{1}$
$\operatorname{sech} x =$	$\pm\frac{\operatorname{cosech} x}{\sqrt{\operatorname{cosech}^2 x + 1}}$	$\operatorname{sech} x$	$\pm\frac{\sqrt{\operatorname{coth}^2 x - 1}}{\operatorname{coth} x}$
$\operatorname{coth} x =$	$\sqrt{\operatorname{cosech}^2 x + 1}$	$\pm\frac{1}{\sqrt{1 - \operatorname{sech}^2 x}}$	$\operatorname{coth} x$

Whenever two signs are shown, choose + sign if x is positive, – sign if x is negative.

Derivatives*

In the following formulas u, v, w represent functions of x , while a, c, n represent fixed real numbers. All arguments in the trigonometric functions are measured in radians, and all inverse trigonometric and hyperbolic functions represent principal values.

$$1. \frac{d}{dx}(a) = 0$$

$$2. \frac{d}{dx}(x) = 1$$

$$3. \frac{d}{dx}(au) = a \frac{du}{dx}$$

$$4. \frac{d}{dx}(u + v - w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$$

$$5. \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$6. \frac{d}{dx}(uvw) = uv \frac{dw}{dx} + vw \frac{du}{dx} + uw \frac{dv}{dx}$$

$$7. \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{1}{v} \frac{du}{dx} - \frac{u}{v^2} \frac{dv}{dx}$$

$$8. \frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$

$$9. \frac{d}{dx}(\sqrt{u}) = -\frac{1}{2\sqrt{u}} \frac{du}{dx}$$

$$10. \frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2} \frac{du}{dx}$$

$$11. \frac{d}{dx}\left(\frac{1}{u^n}\right) = -\frac{n}{u^{n+1}} \frac{du}{dx}$$

$$12. \frac{d}{dx}\left(\frac{u^n}{v^m}\right) = \frac{u^{n-1}}{v^{m+1}} \left(nv \frac{du}{dx} - mu \frac{dv}{dx} \right)$$

$$13. \frac{d}{dx}(u^n v^m) = u^{n-1} v^{m-1} \left(nv \frac{du}{dx} + mu \frac{dv}{dx} \right)$$

$$14. \frac{d}{dx}[f(u)] = \frac{d}{du}[f(u)] \cdot \frac{du}{dx}$$

* Let $y=f(x)$ and $\frac{dy}{dx} = \frac{d[f(x)]}{dx} = f'(x)$ define respectively a function and its derivative for any value x in their common domain. The differential for the function at such a value x is accordingly defined as

$$dy = d[f(x)] = \frac{dy}{dx} dx = \frac{d[f(x)]}{dx} dx = f'(x) dx$$

Each derivative formula has an associated differential formula. For example, formula 6 above has the differential formula

$$d(uvw) = uv dw + vw du + uw dv$$

DERIVATIVES (Continued)

$$15. \frac{d^2}{dx^2}[f(u)] = \frac{df(u)}{du} \cdot \frac{d^2u}{dx^2} + \frac{d^2f(u)}{du^2} \cdot \left(\frac{du}{dx}\right)^2$$

$$16. \frac{d^n}{dx^n}[uv] = \binom{n}{0} v \frac{d^n u}{dx^n} + \binom{n}{1} \frac{dv}{dx} \frac{d^{n-1}u}{dx^{n-1}} + \binom{n}{2} \frac{d^2v}{dx^2} \frac{d^{n-2}u}{dx^{n-2}} \\ + \cdots + \binom{n}{k} \frac{d^k v}{dx^k} \frac{d^{n-k}u}{dx^{n-k}} + \cdots + \binom{n}{n} u \frac{d^n v}{dx^n}$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, the binomial coefficient, n non-negative integer and $\binom{n}{0} = 1$.

$$17. \frac{du}{dx} = \frac{1}{\frac{dx}{du}} \quad \text{if } \frac{dx}{du} \neq 0$$

$$18. \frac{d}{dx}(\log_a u) = (\log_a e) \frac{1}{u} \frac{du}{dx}$$

$$19. \frac{d}{dx}(\log_e u) = \frac{1}{u} \frac{du}{dx}$$

$$20. \frac{d}{dx}(a^u) = a^u (\log_e a) \frac{du}{dx}$$

$$21. \frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

$$22. \frac{d}{dx}(u^v) = v u^{v-1} \frac{du}{dx} + (\log_e u) u^v \frac{dv}{dx}$$

$$23. \frac{d}{dx}(\sin u) = \frac{du}{dx}(\cos u)$$

$$24. \frac{d}{dx}(\cos u) = -\frac{du}{dx}(\sin u)$$

$$25. \frac{d}{dx}(\tan u) = \frac{du}{dx}(\sec^2 u)$$

$$26. \frac{d}{dx}(\cot u) = -\frac{du}{dx}(\csc^2 u)$$

$$27. \frac{d}{dx}(\sec u) = \frac{du}{dx} \sec u \cdot \tan u$$

$$28. \frac{d}{dx}(\csc u) = -\frac{du}{dx} \csc u \cdot \cot u$$

$$29. \frac{d}{dx}(\text{vers } u) = \frac{du}{dx} \sin u$$

$$30. \frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad \left(-\frac{\pi}{2} \leq \arcsin u \leq \frac{\pi}{2}\right)$$

DERIVATIVES (Continued)

31. $\frac{d}{dx}(\arccos u) = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad (0 \leq \arccos u \leq \pi)$
32. $\frac{d}{dx}(\arctan u) = \frac{1}{1+u^2} \frac{du}{dx}, \quad \left(-\frac{\pi}{2} < \arctan u < \frac{\pi}{2}\right)$
33. $\frac{d}{dx}(\operatorname{arccot} u) = -\frac{1}{1+u^2} \frac{du}{dx}, \quad (0 \leq \operatorname{arccot} u \leq \pi)$
34. $\frac{d}{dx}(\operatorname{arcsec} u) = \frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}, \quad \left(0 \leq \operatorname{arcsec} u < \frac{\pi}{2}, -\pi \leq \operatorname{arcsec} u < -\frac{\pi}{2}\right)$
35. $\frac{d}{dx}(\operatorname{arccsc} u) = -\frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}, \quad \left(0 < \operatorname{arccsc} u \leq \frac{\pi}{2}, -\pi < \operatorname{arccsc} u \leq -\frac{\pi}{2}\right)$
36. $\frac{d}{dx}(\operatorname{arccvers} u) = \frac{1}{\sqrt{2u-u^2}} \frac{du}{dx}, \quad (0 \leq \operatorname{arccvers} u \leq \pi)$
37. $\frac{d}{dx}(\sinh u) = \frac{du}{dx}(\cosh u)$
38. $\frac{d}{dx}(\cosh u) = \frac{du}{dx}(\sinh u)$
39. $\frac{d}{dx}(\tanh u) = \frac{du}{dx}(\operatorname{sech}^2 u)$
40. $\frac{d}{dx}(\coth u) = -\frac{du}{dx}(\operatorname{csch}^2 u)$
41. $\frac{d}{dx}(\operatorname{sech} u) = -\frac{du}{dx}(\operatorname{sech} u \cdot \tanh u)$
42. $\frac{d}{dx}(\operatorname{csch} u) = -\frac{du}{dx}(\operatorname{csch} u \cdot \coth u)$
43. $\frac{d}{dx}(\sinh^{-1} u) = \frac{d}{dx}[\log(u + \sqrt{u^2+1})] = \frac{1}{\sqrt{u^2+1}} \frac{du}{dx}$
44. $\frac{d}{dx}(\cosh^{-1} u) = \frac{d}{dx}[\log(u + \sqrt{u^2-1})] = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx}, \quad (u > 1, \cosh^{-1} u > 0)$
45. $\frac{d}{dx}(\tanh^{-1} u) = \frac{d}{dx}\left[\frac{1}{2} \log \frac{1+u}{1-u}\right] = \frac{1}{1-u^2} \frac{du}{dx}, \quad (u^2 < 1)$
46. $\frac{d}{dx}(\coth^{-1} u) = \frac{d}{dx}\left[\frac{1}{2} \log \frac{u+1}{u-1}\right] = \frac{1}{1-u^2} \frac{du}{dx}, \quad (u^2 > 1)$
47. $\frac{d}{dx}(\operatorname{sech}^{-1} u) = \frac{d}{dx}\left[\log \frac{1+\sqrt{1-u^2}}{u}\right] = -\frac{1}{u\sqrt{1-u^2}} \frac{du}{dx}, \quad (0 < u < 1, \operatorname{sech}^{-1} u > 0)$
48. $\frac{d}{dx}(\operatorname{csch}^{-1} u) = \frac{d}{dx}\left[\log \frac{1+\sqrt{1+u^2}}{u}\right] = -\frac{1}{|u|\sqrt{1+u^2}} \frac{du}{dx}$

$$49. \quad \frac{d}{dq} \int_p^q f(x) dx = f(q), \quad [p \text{ constant}]$$

$$50. \quad \frac{d}{dp} \int_p^q f(x) dx = -f(p), \quad [q \text{ constant}]$$

$$51. \quad \frac{d}{da} \int_p^q f(x, a) dx = \int_p^q \frac{\partial}{\partial a} [f(x, a)] dx + f(q, a) \frac{dq}{da} - f(p, a) \frac{dp}{da}$$

INTEGRATION

The following is a brief discussion of some integration techniques. A more complete discussion can be found in a number of good text books. However, the purpose of this introduction is simply to discuss a few of the important techniques which may be used, in conjunction with the integral table which follows, to integrate particular functions.

No matter how extensive the integral table, it is a fairly uncommon occurrence to find in the table the exact integral desired. Usually some form of transformation will have to be made. The simplest type of transformation, and yet the most general, is substitution. Simple forms of substitution, such as $y = ax$, are employed almost unconsciously by experienced users of integral tables. Other substitutions may require more thought. In some sections of the tables, appropriate substitutions are suggested for integrals which are similar to, but not exactly like, integrals in the table. Finding the right substitution is largely a matter of intuition and experience.

Several precautions must be observed when using substitutions:

1. Be sure to make the substitution in the dx term, as well as everywhere else in the integral.
2. Be sure that the function substituted is one-to-one and continuous. If this is not the case, the integral must be restricted in such a way as to make it true. See the example following.
3. With definite integrals, the limits should also be expressed in terms of the new dependent variable. With indefinite integrals, it is necessary to perform the reverse substitution to obtain the answer in terms of the original independent variable. This may also be done for definite integrals, but it is usually easier to change the limits.

Example:

$$\int \frac{x^4}{\sqrt{a^2 - x^2}} dx$$

Here we make the substitution $x = |a| \sin \theta$. Then $dx = |a| \cos \theta d\theta$, and

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = |a| \sqrt{1 - \sin^2 \theta} = |a| \cos \theta$$

Notice the absolute value signs. It is very important to keep in mind that a square root radical always denotes the positive square root, and to assure the sign is always kept positive. Thus $\sqrt{x^2} = |x|$. Failure to observe this is a common cause of errors in integration.

Notice also that the indicated substitution is not a one-to-one function, that is, it does not have a unique inverse. Thus we must restrict the range of θ in such a way as to make the function one-to-one. Fortunately, this is easily done by solving for θ

$$\theta = \sin^{-1} \frac{x}{|a|}$$

and restricting the inverse sine to the principal values, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

Thus the integral becomes

$$\int \frac{a^4 \sin^4 \theta |a| \cos \theta d\theta}{|a| |\cos \theta|}$$

Now, however, in the range of values chosen for θ , $\cos \theta$ is always positive. Thus we may remove the absolute value signs from $\cos \theta$ in the denominator. (This is one of the reasons that the principal values of the inverse trigonometric functions are defined as they are.)

Then the $\cos \theta$ terms cancel, and the integral becomes

$$a^4 \int \sin^4 \theta d\theta$$

By application of integral formulas 299 and 296, we integrate this to

$$-a^4 \frac{\sin^3 \theta \cos \theta}{4} - \frac{3a^4}{8} \cos \theta \sin \theta + \frac{3a^4}{8} \theta + C$$

We now must perform the inverse substitution to get the result in terms of x . We have

$$\theta = \sin^{-1} \frac{x}{|a|}$$

$$\sin \theta = \frac{x}{|a|}$$

Then

$$\cos \theta = \pm \sqrt{1 - \sin^2 \theta} = \pm \sqrt{1 - \frac{x^2}{a^2}} = \pm \frac{\sqrt{a^2 - x^2}}{|a|}.$$

Because of the previously mentioned fact that $\cos \theta$ is positive, we may omit the $\pm$ sign. The reverse substitution then produces the final answer

$$\int \frac{x^4}{\sqrt{a^2 - x^2}} dx = -\frac{1}{4}x^3 \sqrt{a^2 - x^2} - \frac{3}{8}a^2 x \sqrt{a^2 - x^2} + \frac{3a^4}{8} \sin^{-1} \frac{x}{|a|} + C.$$

Any rational function of x may be integrated, if the denominator is factored into linear and irreducible quadratic factors. The function may then be broken into partial fractions, and the individual partial fractions integrated by use of the appropriate formula from the integral table. See the section on partial fractions for further information.

Many integrals may be reduced to rational functions by proper substitutions. For example,

$$z = \tan \frac{x}{2}$$

will reduce any rational function of the six trigonometric functions of x to a rational function of z . (Frequently there are other substitutions which are simpler to use, but this one will always work. See integral formula number 484.)

Any rational function of x and $\sqrt{ax+b}$ may be reduced to a rational function of z by making the substitution

$$z = \sqrt{ax+b}.$$

Other likely substitutions will be suggested by looking at the form of the integrand.

The other main method of transforming integrals is integration by parts. This involves applying formula number 5 or 6 in the accompanying integral table. The critical factor in this method is the choice of the functions u and v . In order for the method to be successful, $v = \int dv$ and $\int v du$ must be easier to integrate than the original integral. Again, this choice is largely a matter of intuition and experience.

Example:

$$\int x \sin x \, dx$$

Two obvious choices are $u = x$, $dv = \sin x \, dx$, or $u = \sin x$, $dv = x \, dx$. Since a preliminary mental calculation indicates that $\int v \, du$ in the second choice would be more, rather than less, complicated than the original integral (it would contain x^2), we use the first choice.

$$\begin{aligned} u &= x & du &= dx \\ dv &= \sin x \, dx & v &= -\cos x \\ \int x \sin x \, dx &= \int u \, dv = uv - \int v \, du = -x \cos x + \int \cos x \, dx \\ &= \sin x - x \cos x \end{aligned}$$

Of course, this result could have been obtained directly from the integral table, but it provides a simple example of the method. In more complicated examples the choice of u and v may not be so obvious, and several different choices may have to be tried. Of course, there is no guarantee that any of them will work.

Integration by parts may be applied more than once, or combined with substitution. A fairly common case is illustrated by the following example.

Example:

$$\int e^x \sin x \, dx$$

Let

$$\begin{aligned} u &= e^x & \text{Then } du &= e^x \, dx \\ dv &= \sin x \, dx & v &= -\cos x \\ \int e^x \sin x \, dx &= \int u \, dv = uv - \int v \, du = -e^x \cos x + \int e^x \cos x \, dx \end{aligned}$$

In this latter integral,

$$\begin{aligned} \text{Let } u &= e^x & \text{Then } du &= e^x \, dx \\ dv &= \cos x \, dx & v &= \sin x \\ \int e^x \sin x \, dx &= -e^x \cos x + \int e^x \cos x \, dx = -e^x \cos x + \int u \, dv \\ &= -e^x \cos x + uv - \int v \, du \\ &= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx \end{aligned}$$

This looks as if a circular transformation has taken place, since we are back at the same integral we started from. However, the above equation can be solved algebraically for the required integral:

$$\int e^x \sin x \, dx = \frac{1}{2}(e^x \sin x - e^x \cos x)$$

In the second integration by parts, if the parts had been chosen as $u = \cos x$, $dv = e^x \, dx$, we would indeed have made a circular transformation, and returned to the starting place.

In general, when doing repeated integration by parts, one should never choose the function u at any stage to be the same as the function v at the previous stage, or a constant times the previous v .

The following rule is called the extended rule for integration by parts. It is the result of $n + 1$ successive applications of integration by parts.

If

$$\begin{aligned} g_1(x) &= \int g(x) dx, & g_2(x) &= \int g_1(x) dx, \\ g_3(x) &= \int g_2(x) dx, \dots, & g_m(x) &= \int g_{m-1}(x) dx, \dots, \end{aligned}$$

then

$$\begin{aligned} \int f(x) \cdot g(x) dx &= f(x) \cdot g_1(x) - f'(x) \cdot g_2(x) + f''(x) \cdot g_3(x) - + \dots \\ &+ (-1)^n f^{(n)}(x) g_{n+1}(x) + (-1)^{n+1} \int f^{(n+1)}(x) g_{n+1}(x) dx. \end{aligned}$$

A useful special case of the above rule is when $f(x)$ is a polynomial of degree n . Then $f^{(n+1)}(x) = 0$, and

$$\int f(x) \cdot g(x) dx = f(x) \cdot g_1(x) - f'(x) \cdot g_2(x) + f''(x) \cdot g_3(x) - + \dots + (-1)^n f^{(n)}(x) g_{n+1}(x) + C$$

Example:

If $f(x) = x^2$, $g(x) = \sin x$

$$\int x^2 \sin x dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

Another application of this formula occurs if

$$f''(x) = af(x) \quad \text{and} \quad g''(x) = bg(x),$$

where a and b are unequal constants. In this case, by a process similar to that used in the above example for $\int e^x \sin x dx$, we get the formula

$$\int f(x)g(x) dx = \frac{f(x) \cdot g'(x) - f'(x) \cdot g(x)}{b - a} + C$$

This formula could have been used in the example mentioned. Here is another example.

Example:

If $f(x) = e^{2x}$, $g(x) = \sin 3x$, then $a = 4$, $b = -9$, and

$$\int e^{2x} \sin 3x dx = \frac{3 e^{2x} \cos 3x - 2 e^{2x} \sin 3x}{-9 - 4} + C = \frac{e^{2x}}{13} (2 \sin 3x - 3 \cos 3x) + C$$

The following additional points should be observed when using this table.

1. A constant of integration is to be supplied with the answers for indefinite integrals.
2. Logarithmic expressions are to base $e = 2.71828 \dots$, unless otherwise specified, and are to be evaluated for the absolute value of the arguments involved therein.
3. All angles are measured in radians, and inverse trigonometric and hyperbolic functions represent principal values, unless otherwise indicated.
4. If the application of a formula produces either a zero denominator or the square root of a negative number in the result, there is usually available another form of the answer which

avoids this difficulty. In many of the results, the excluded values are specified, but when such are omitted it is presumed that one can tell what these should be, especially when difficulties of the type herein mentioned are obtained.

5. When inverse trigonometric functions occur in the integrals, be sure that any replacements made for them are strictly in accordance with the rules for such functions. This causes little difficulty when the argument of the inverse trigonometric function is positive, since then all angles involved are in the first quadrant. However, if the argument is negative, special care must be used. Thus if $u > 0$,

$$\sin^{-1} u = \cos^{-1} \sqrt{1 - u^2} = \csc^{-1} \frac{1}{u}, \text{ etc.}$$

However, if $u < 0$,

$$\sin^{-1} u = -\cos^{-1} \sqrt{1 - u^2} = -\pi - \csc^{-1} \frac{1}{u}, \text{ etc.}$$

See the section on inverse trigonometric functions for a full treatment of the allowable substitutions.

6. In integrals 340–345 and some others, the right side includes expressions of the form

$$A \tan^{-1}[B + C \tan f(x)].$$

In these formulas, the $\tan^{-1}$ does not necessarily represent the principal value. Instead of always employing the principal branch of the inverse tangent function, one must instead use that branch of the inverse tangent function upon which $f(x)$ lies for any particular choice of x .

Example:

$$\begin{aligned} \int_0^{4\pi} \frac{dx}{2 + \sin x} &= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2 \tan \frac{x}{2} + 1}{\sqrt{3}} \Bigg|_0^{4\pi} \\ &= \frac{2}{\sqrt{3}} \left[\tan^{-1} \frac{2 \tan 2\pi + 1}{\sqrt{3}} - \tan^{-1} \frac{2 \tan 0 + 1}{\sqrt{3}} \right] \\ &= \frac{2}{\sqrt{3}} \left[\frac{13\pi}{6} - \frac{\pi}{6} \right] = \frac{4\pi}{\sqrt{3}} = \frac{4\sqrt{3}\pi}{3} \end{aligned}$$

Here

$$\tan^{-1} \frac{2 \tan 2\pi + 1}{\sqrt{3}} = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{13\pi}{6},$$

since $f(x) = 2\pi$; and

$$\tan^{-1} \frac{2 \tan 0 + 1}{\sqrt{3}} = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6},$$

since $f(x) = 0$.

7. B_n and E_n where used in Integrals represents the Bernoulli and Euler numbers as defined in tables of Bernoulli and Euler polynomials contained in certain mathematics reference and hand-books, as for example, Beyer, W. H., *Handbook of Mathematical Sciences*, 5th ed., CRC Press, Inc., West Palm Beach 1978, 577–583.

INTEGRALS

ELEMENTARY FORMS

$$1. \int a \, dx = ax$$

$$2. \int a \cdot f(x) \, dx = a \int f(x) \, dx$$

$$3. \int \phi(y) \, dx = \int \frac{\phi(y)}{y'} \, dy, \quad \text{where } y' = \frac{dy}{dx}$$

$$4. \int (u + v) \, dx = \int u \, dx + \int v \, dx, \quad \text{where } u \text{ and } v \text{ are any functions of } x$$

$$5. \int u \, dv = u \int dv - \int v \, du = uv - \int v \, du$$

$$6. \int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$$

$$7. \int x^n \, dx = \frac{x^{n+1}}{n+1}, \quad \text{except } n = -1$$

$$8. \int \frac{f'(x) \, dx}{f(x)} = \log f(x), \quad (df(x) = f'(x) \, dx)$$

$$9. \int \frac{dx}{x} = \log x$$

$$10. \int \frac{f'(x) \, dx}{2\sqrt{f(x)}} = \sqrt{f(x)}, \quad (df(x) = f'(x) \, dx)$$

$$11. \int e^x \, dx = e^x$$

$$12. \int e^{ax} \, dx = e^{ax}/a$$

$$13. \int b^{ax} \, dx = \frac{b^{ax}}{a \log b}, \quad (b > 0)$$

$$14. \int \log x \, dx = x \log x - x$$

$$15. \int a^x \log a \, dx = a^x, \quad (a > 0)$$

$$16. \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$17. \int \frac{dx}{a^2 - x^2} = \begin{cases} \frac{1}{a} \tanh^{-1} \frac{x}{a} \\ \text{or} \\ \frac{1}{2a} \log \frac{a+x}{a-x}, \end{cases} \quad (a^2 > x^2)$$

$$18. \int \frac{dx}{x^2 - a^2} = \begin{cases} -\frac{1}{a} \coth^{-1} \frac{x}{a} \\ \text{or} \\ \frac{1}{2a} \log \frac{x-a}{x+a}, \end{cases} \quad (x^2 > a^2)$$

INTEGRALS (Continued)

$$19. \int \frac{dx}{\sqrt{a^2 - x^2}} = \begin{cases} \sin^{-1} \frac{x}{|a|} \\ \text{or} \\ -\cos^{-1} \frac{x}{|a|}, \end{cases} \quad (a^2 > x^2)$$

$$20. \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2})$$

$$21. \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|} \sec^{-1} \frac{x}{a}$$

$$22. \int \frac{dx}{x\sqrt{a^2 \pm x^2}} = -\frac{1}{a} \log \left(\frac{a + \sqrt{a^2 \pm x^2}}{x} \right)$$

FORMS CONTAINING $(a + bx)$

For forms containing $a + bx$, but not listed in the table, the substitution $u = \frac{a + bx}{x}$ may prove helpful.

$$23. \int (a + bx)^n dx = \frac{(a + bx)^{n+1}}{(n+1)b}, \quad (n \neq -1)$$

$$24. \int x(a + bx)^n dx = \frac{1}{b^2(n+2)}(a + bx)^{n+2} - \frac{a}{b^2(n+1)}(a + bx)^{n+1}, \quad (n \neq -1, -2)$$

$$25. \int x^2(a + bx)^n dx = \frac{1}{b^3} \left[\frac{(a + bx)^{n+3}}{n+3} - 2a \frac{(a + bx)^{n+2}}{n+2} + a^2 \frac{(a + bx)^{n+1}}{n+1} \right]$$

$$26. \int x^m(a + bx)^n dx = \begin{cases} \frac{x^{m+1}(a + bx)^n}{m+n+1} + \frac{an}{m+n+1} \int x^m(a + bx)^{n-1} dx \\ \text{or} \\ \frac{1}{a(n+1)} \left[-x^{m+1}(a + bx)^{n+1} + (m+n+2) \int x^m(a + bx)^{n+1} dx \right] \\ \text{or} \\ \frac{1}{b(m+n+1)} \left[x^m(a + bx)^{n+1} - ma \int x^{m-1}(a + bx)^n dx \right] \end{cases}$$

$$27. \int \frac{dx}{a + bx} = \frac{1}{b} \log(a + bx)$$

$$28. \int \frac{dx}{(a + bx)^2} = -\frac{1}{b(a + bx)}$$

$$29. \int \frac{dx}{(a + bx)^3} = -\frac{1}{2b(a + bx)^2}$$

$$30. \int \frac{x dx}{a + bx} = \begin{cases} \frac{1}{b^2} [a + bx - a \log(a + bx)] \\ \text{or} \\ \frac{x}{b} - \frac{a}{b^2} \log(a + bx) \end{cases}$$

INTEGRALS (Continued)

31. $\int \frac{x dx}{(a+bx)^2} = \frac{1}{b^2} \left[\log(a+bx) + \frac{a}{a+bx} \right]$
32. $\int \frac{x dx}{(a+bx)^n} = \frac{1}{b^2} \left[\frac{-1}{(n-2)(a+bx)^{n-2}} + \frac{a}{(n-1)(a+bx)^{n-1}} \right], \quad n \neq 1, 2$
33. $\int \frac{x^2 dx}{a+bx} = \frac{1}{b^3} \left[\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2 \log(a+bx) \right]$
34. $\int \frac{x^2 dx}{(a+bx)^2} = \frac{1}{b^3} \left[a+bx - 2a \log(a+bx) - \frac{a^2}{a+bx} \right]$
35. $\int \frac{x^2 dx}{(a+bx)^3} = \frac{1}{b^3} \left[\log(a+bx) + \frac{2a}{a+bx} - \frac{a^2}{2(a+bx)^2} \right]$
36. $\int \frac{x^2 dx}{(a+bx)^n} = \frac{1}{b^3} \left[\frac{-1}{(n-3)(a+bx)^{n-3}} + \frac{2a}{(n-2)(a+bx)^{n-2}} - \frac{a^2}{(n-1)(a+bx)^{n-1}} \right], \quad n \neq 1, 2, 3$
37. $\int \frac{dx}{x(a+bx)} = -\frac{1}{a} \log \frac{a+bx}{x}$
38. $\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2} \log \frac{a+bx}{x}$
39. $\int \frac{dx}{x(a+bx)^3} = \frac{1}{a^3} \left[\frac{1}{2} \left(\frac{2a+bx}{a+bx} \right)^2 + \log \frac{x}{a+bx} \right]$
40. $\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2} \log \frac{a+bx}{x}$
41. $\int \frac{dx}{x^3(a+bx)} = \frac{2bx-a}{2a^2x^2} + \frac{b^2}{a^3} \log \frac{x}{a+bx}$
42. $\int \frac{dx}{x^2(a+bx)^2} = -\frac{a+2bx}{a^2x(a+bx)} + \frac{2b}{a^3} \log \frac{a+bx}{x}$

FORMS CONTAINING $c^2 \pm x^2$, $x^2 - c^2$

43. $\int \frac{dx}{c^2+x^2} = \frac{1}{c} \tan^{-1} \frac{x}{c}$
44. $\int \frac{dx}{c^2-x^2} = \frac{1}{2c} \log \frac{c+x}{c-x}, \quad (c^2 > x^2)$
45. $\int \frac{dx}{x^2-c^2} = \frac{1}{2c} \log \frac{x-c}{x+c}, \quad (x^2 > c^2)$
46. $\int \frac{x dx}{c^2 \pm x^2} = \pm \frac{1}{2} \log(c^2 \pm x^2)$
47. $\int \frac{x dx}{(c^2 \pm x^2)^{n+1}} = \mp \frac{1}{2n(c^2 \pm x^2)^n}$
48. $\int \frac{dx}{(c^2 \pm x^2)^n} = \frac{1}{2c^2(n-1)} \left[\frac{x}{(c^2 \pm x^2)^{n-1}} + (2n-3) \int \frac{dx}{(c^2 \pm x^2)^{n-1}} \right]$
49. $\int \frac{dx}{(x^2-c^2)^n} = \frac{1}{2c^2(n-1)} \left[-\frac{x}{(x^2-c^2)^{n-1}} - (2n-3) \int \frac{dx}{(x^2-c^2)^{n-1}} \right]$
50. $\int \frac{x dx}{x^2-c^2} = \frac{1}{2} \log(x^2-c^2)$

INTEGRALS (Continued)

$$51. \int \frac{x dx}{(x^2 - c^2)^{n+1}} = -\frac{1}{2n(x^2 - c^2)^n}$$

FORMS CONTAINING $a + bx$ and $c + dx$

$$u = a + bx, \quad v = c + dx, \quad k = ad - bc$$

If $k = 0$, then $v = \frac{c}{a}u$

$$52. \int \frac{dx}{u \cdot v} = \frac{1}{k} \cdot \log \left(\frac{v}{u} \right)$$

$$53. \int \frac{x dx}{u \cdot v} = \frac{1}{k} \left[\frac{a}{b} \log(u) - \frac{c}{d} \log(v) \right]$$

$$54. \int \frac{dx}{u^2 \cdot v} = \frac{1}{k} \left(\frac{1}{u} + \frac{d}{k} \log \frac{v}{u} \right)$$

$$55. \int \frac{x dx}{u^2 \cdot v} = \frac{-a}{bku} - \frac{c}{k^2} \log \frac{v}{u}$$

$$56. \int \frac{x^2 dx}{u^2 \cdot v} = \frac{a^2}{b^2ku} + \frac{1}{k^2} \left[\frac{c^2}{d} \log(v) + \frac{a(k - bc)}{b^2} \log(u) \right]$$

$$57. \int \frac{dx}{u^m \cdot v^n} = \frac{1}{k(m-1)} \left[\frac{-1}{u^{m-1} \cdot v^{n-1}} - (m+n-2)b \int \frac{dx}{u^m \cdot v^{n-1}} \right]$$

$$58. \int \frac{u}{v} dx = \frac{bx}{d} + \frac{k}{d^2} \log(v)$$

$$59. \int \frac{u^m dx}{v^n} = \begin{cases} \frac{-1}{k(n-1)} \left[\frac{u^{m+1}}{v^{n-1}} + b(n-m-2) \int \frac{u^m}{v^{n-1}} dx \right] \\ \text{or} \\ \frac{-1}{d(n-m-1)} \left[\frac{u^m}{v^{n-1}} + mk \int \frac{u^{m-1}}{v^n} dx \right] \\ \text{or} \\ \frac{-1}{d(n-1)} \left[\frac{u^m}{v^{n-1}} - mb \int \frac{u^{m-1}}{v^{n-1}} dx \right] \end{cases}$$

FORMS CONTAINING $(a + bx^n)$

$$60. \int \frac{dx}{a + bx^2} = \frac{1}{\sqrt{ab}} \tan^{-1} \frac{x\sqrt{ab}}{a}, \quad (ab > 0)$$

$$61. \int \frac{dx}{a + bx^2} = \begin{cases} \frac{1}{2\sqrt{-ab}} \log \frac{a + x\sqrt{-ab}}{a - x\sqrt{-ab}}, & (ab < 0) \\ \text{or} \\ \frac{1}{\sqrt{-ab}} \tanh^{-1} \frac{x\sqrt{-ab}}{a}, & (ab < 0) \end{cases}$$

$$62. \int \frac{dx}{a^2 + b^2x^2} = \frac{1}{ab} \tan^{-1} \frac{bx}{a}$$

$$63. \int \frac{x dx}{a + bx^2} = \frac{1}{2b} \log(a + bx^2)$$

$$64. \int \frac{x^2 dx}{a + bx^2} = \frac{x}{b} - \frac{a}{b} \int \frac{dx}{a + bx^2}$$

$$65. \int \frac{dx}{(a + bx^2)^2} = \frac{x}{2a(a + bx^2)} + \frac{1}{2a} \int \frac{dx}{a + bx^2}$$

INTEGRALS (Continued)

$$66. \int \frac{dx}{a^2 - b^2 x^2} = \frac{1}{2ab} \log \frac{a + bx}{a - bx}$$

$$67. \int \frac{dx}{(a + bx^2)^{m+1}} = \begin{cases} \frac{1}{2ma(a + bx^2)^m} + \frac{2m-1}{2ma} \int \frac{dx}{(a + bx^2)^m} \\ \text{or} \\ \frac{(2m)!}{(m!)^2} \left[\frac{x}{2a} \sum_{r=1}^m \frac{r!(r-1)!}{(4a)^{m-r}(2r)!(a + bx^2)^r} + \frac{1}{(4a)^m} \int \frac{dx}{a + bx^2} \right] \end{cases}$$

$$68. \int \frac{x dx}{(a + bx^2)^{m+1}} = -\frac{1}{2bm(a + bx^2)^m}$$

$$69. \int \frac{x^2 dx}{(a + bx^2)^{m+1}} = \frac{-x}{2mb(a + bx^2)^m} + \frac{1}{2mb} \int \frac{dx}{(a + bx^2)^m}$$

$$70. \int \frac{dx}{x(a + bx^2)} = \frac{1}{2a} \log \frac{x^2}{a + bx^2}$$

$$71. \int \frac{dx}{x^2(a + bx^2)} = -\frac{1}{ax} - \frac{b}{a} \int \frac{dx}{a + bx^2}$$

$$72. \int \frac{dx}{x(a + bx^2)^{m+1}} = \begin{cases} \frac{1}{2am(a + bx^2)^m} + \frac{1}{a} \int \frac{dx}{x(a + bx^2)^m} \\ \text{or} \\ \frac{1}{2a^{m+1}} \left[\sum_{r=1}^m \frac{a^r}{r(a + bx^2)^r} + \log \frac{x^2}{a + bx^2} \right] \end{cases}$$

$$73. \int \frac{dx}{x^2(a + bx^2)^{m+1}} = \frac{1}{a} \int \frac{dx}{x^2(a + bx^2)^m} - \frac{b}{a} \int \frac{dx}{(a + bx^2)^{m+1}}$$

$$74. \int \frac{dx}{a + bx^3} = \frac{k}{3a} \left[\frac{1}{2} \log \frac{(k+x)^3}{a + bx^3} + \sqrt{3} \tan^{-1} \frac{2x-k}{k\sqrt{3}} \right], \quad \left(k = \sqrt[3]{\frac{a}{b}} \right)$$

$$75. \int \frac{x dx}{a + bx^3} = \frac{1}{3bk} \left[\frac{1}{2} \log \frac{a + bx^3}{(k+x)^3} + \sqrt{3} \tan^{-1} \frac{2x-k}{k\sqrt{3}} \right], \quad \left(k = \sqrt[3]{\frac{a}{b}} \right)$$

$$76. \int \frac{x^2 dx}{a + bx^3} = \frac{1}{3b} \log(a + bx^3)$$

$$77. \int \frac{dx}{a + bx^4} = \frac{k}{2a} \left[\frac{1}{2} \log \frac{x^2 + 2kx + 2k^2}{x^2 - 2kx + 2k^2} + \tan^{-1} \frac{2kx}{2k^2 - x^2} \right], \quad \left(ab > 0, k = \sqrt[4]{\frac{a}{4b}} \right)$$

$$78. \int \frac{dx}{a + bx^4} = \frac{k}{2a} \left[\frac{1}{2} \log \frac{x+k}{x-k} + \tan^{-1} \frac{x}{k} \right], \quad \left(ab < 0, k = \sqrt[4]{-\frac{a}{b}} \right)$$

$$79. \int \frac{x dx}{a + bx^4} = \frac{1}{2bk} \tan^{-1} \frac{x^2}{k}, \quad \left(ab > 0, k = \sqrt[4]{\frac{a}{b}} \right)$$

$$80. \int \frac{x dx}{a + bx^4} = \frac{1}{4bk} \log \frac{x^2 - k}{x^2 + k}, \quad \left(ab < 0, k = \sqrt[4]{-\frac{a}{b}} \right)$$

$$81. \int \frac{x^2 dx}{a + bx^4} = \frac{1}{4bk} \left[\frac{1}{2} \log \frac{x^2 - 2kx + 2k^2}{x^2 + 2kx + 2k^2} + \tan^{-1} \frac{2kx}{2k^2 - x^2} \right], \quad \left(ab > 0, k = \sqrt[4]{\frac{a}{4b}} \right)$$

INTEGRALS (Continued)

$$82. \int \frac{x^2 dx}{a + bx^4} = \frac{1}{4bk} \left[\log \frac{x-k}{x+k} + 2 \tan^{-1} \frac{x}{k} \right], \quad \left(ab < 0, k = \sqrt[4]{-\frac{a}{b}} \right)$$

$$83. \int \frac{x^3 dx}{a + bx^4} = \frac{1}{4b} \log(a + bx^4)$$

$$84. \int \frac{dx}{x(a + bx^n)} = \frac{1}{an} \log \frac{x^n}{a + bx^n}$$

$$85. \int \frac{dx}{(a + bx^n)^{m+1}} = \frac{1}{a} \int \frac{dx}{(a + bx^n)^m} - \frac{b}{a} \int \frac{x^n dx}{(a + bx^n)^{m+1}}$$

$$86. \int \frac{x^m dx}{(a + bx^n)^{p+1}} = \frac{1}{b} \int \frac{x^{m-n} dx}{(a + bx^n)^p} - \frac{a}{b} \int \frac{x^{m-n} dx}{(a + bx^n)^{p+1}}$$

$$87. \int \frac{dx}{x^m(a + bx^n)^{p+1}} = \frac{1}{a} \int \frac{dx}{x^m(a + bx^n)^p} - \frac{b}{a} \int \frac{dx}{x^{m-n}(a + bx^n)^{p+1}}$$

$$88. \int x^m(a + bx^n)^p dx = \begin{cases} \frac{1}{b(np + m + 1)} \left[x^{m-n+1}(a + bx^n)^{p+1} - a(m - n + 1) \int x^{m-n}(a + bx^n)^p dx \right] \\ \text{or} \\ \frac{1}{np + m + 1} \left[x^{m+1}(a + bx^n)^p + anp \int x^m(a + bx^n)^{p-1} dx \right] \\ \text{or} \\ \frac{1}{a(m + 1)} \left[x^{m+1}(a + bx^n)^{p+1} - (m + 1 + np + n)b \int x^{m+n}(a + bx^n)^p dx \right] \\ \text{or} \\ \frac{1}{an(p + 1)} \left[-x^{m+1}(a + bx^n)^{p+1} + (m + 1 + np + n) \int x^m(a + bx^n)^{p+1} dx \right] \end{cases}$$

FORMS CONTAINING $c^3 \pm x^3$

$$89. \int \frac{dx}{c^3 \pm x^3} = \pm \frac{1}{6c^2} \log \frac{(c \pm x)^3}{c^3 \pm x^3} + \frac{1}{c^2\sqrt{3}} \tan^{-1} \frac{2x \mp c}{c\sqrt{3}}$$

$$90. \int \frac{dx}{(c^3 \pm x^3)^2} = \frac{x}{3c^3(c^3 \pm x^3)} + \frac{2}{3c^3} \int \frac{dx}{c^3 \pm x^3}$$

$$91. \int \frac{dx}{(c^3 \pm x^3)^{n+1}} = \frac{1}{3nc^3} \left[\frac{x}{(c^3 \pm x^3)^n} + (3n - 1) \int \frac{dx}{(c^3 \pm x^3)^n} \right]$$

$$92. \int \frac{x dx}{c^3 \pm x^3} = \frac{1}{6c} \log \frac{c^3 \pm x^3}{(c \pm x)^3} \pm \frac{1}{c\sqrt{3}} \tan^{-1} \frac{2x \mp c}{c\sqrt{3}}$$

$$93. \int \frac{x dx}{(c^3 \pm x^3)^2} = \frac{x^2}{3c^3(c^3 \pm x^3)} + \frac{1}{3c^3} \int \frac{x dx}{c^3 \pm x^3}$$

INTEGRALS (Continued)

94. $\int \frac{x dx}{(c^3 \pm x^3)^{n+1}} = \frac{1}{3nc^3} \left[\frac{x^2}{(c^3 \pm x^3)^n} + (3n-2) \int \frac{x dx}{(c^3 \pm x^3)^n} \right]$
95. $\int \frac{x^2 dx}{c^3 \pm x^3} = \pm \frac{1}{3} \log(c^3 \pm x^3)$
96. $\int \frac{x^2 dx}{(c^3 \pm x^3)^{n+1}} = \mp \frac{1}{3n(c^3 \pm x^3)^n}$
97. $\int \frac{dx}{x(c^3 \pm x^3)} = \frac{1}{3c^3} \log \frac{x^3}{c^3 \pm x^3}$
98. $\int \frac{dx}{x(c^3 \pm x^3)^2} = \frac{1}{3c^3(c^3 \pm x^3)} + \frac{1}{3c^6} \log \frac{x^3}{c^3 \pm x^3}$
99. $\int \frac{dx}{x(c^3 \pm x^3)^{n+1}} = \frac{1}{3nc^3(c^3 \pm x^3)^n} + \frac{1}{c^3} \int \frac{dx}{x(c^3 \pm x^3)^n}$
100. $\int \frac{dx}{x^2(c^3 \pm x^3)} = -\frac{1}{c^3 x} \mp \frac{1}{c^3} \int \frac{x dx}{c^3 \pm x^3}$
101. $\int \frac{dx}{x^2(c^3 \pm x^3)^{n+1}} = \frac{1}{c^3} \int \frac{dx}{x^2(c^3 \pm x^3)^n} \mp \frac{1}{c^3} \int \frac{x dx}{(c^3 \pm x^3)^{n+1}}$

FORMS CONTAINING $c^4 \pm x^4$

102. $\int \frac{dx}{c^4 + x^4} = \frac{1}{2c^3\sqrt{2}} \left[\frac{1}{2} \log \frac{x^2 + cx\sqrt{2} + c^2}{x^2 - cx\sqrt{2} + c^2} + \tan^{-1} \frac{cx\sqrt{2}}{c^2 - x^2} \right]$
103. $\int \frac{dx}{c^4 - x^4} = \frac{1}{2c^3} \left[\frac{1}{2} \log \frac{c+x}{c-x} + \tan^{-1} \frac{x}{c} \right]$
104. $\int \frac{x dx}{c^4 + x^4} = \frac{1}{2c^2} \tan^{-1} \frac{x^2}{c^2}$
105. $\int \frac{x dx}{c^4 - x^4} = \frac{1}{4c^2} \log \frac{c^2 + x^2}{c^2 - x^2}$
106. $\int \frac{x^2 dx}{c^4 + x^4} = \frac{1}{2c\sqrt{2}} \left[\frac{1}{2} \log \frac{x^2 - cx\sqrt{2} + c^2}{x^2 + cx\sqrt{2} + c^2} + \tan^{-1} \frac{cx\sqrt{2}}{c^2 - x^2} \right]$
107. $\int \frac{x^2 dx}{c^4 - x^4} = \frac{1}{2c} \left[\frac{1}{2} \log \frac{c+x}{c-x} - \tan^{-1} \frac{x}{c} \right]$
108. $\int \frac{x^3 dx}{c^4 \pm x^4} = \pm \frac{1}{4} \log(c^4 \pm x^4)$

FORMS CONTAINING $(a + bx + cx^2)$

$$X = a + bx + cx^2 \text{ and } q = 4ac - b^2$$

If $q=0$, then $X = c\left(x + \frac{b}{2c}\right)^2$, and formulas starting with 23 should be used in place of these.

109. $\int \frac{dx}{X} = \frac{2}{\sqrt{q}} \tan^{-1} \frac{2cx+b}{\sqrt{q}}, \quad (q > 0)$
110. $\int \frac{dx}{X} = \begin{cases} \frac{-2}{\sqrt{-q}} \tanh^{-1} \frac{2cx+b}{\sqrt{-q}} \\ \text{or} \\ \frac{1}{\sqrt{-q}} \log \frac{2cx+b-\sqrt{-q}}{2cx+b+\sqrt{-q}}, \end{cases} \quad (q < 0)$
111. $\int \frac{dx}{X^2} = \frac{2cx+b}{qX} + \frac{2c}{q} \int \frac{dx}{X}$

INTEGRALS (Continued)

$$112. \int \frac{dx}{X^3} = \frac{2cx+b}{q} \left(\frac{1}{2X^2} + \frac{3c}{qX} \right) + \frac{6c^2}{q^2} \int \frac{dx}{X}$$

$$113. \int \frac{dx}{X^{n+1}} = \begin{cases} \frac{2cx+b}{nqX^n} + \frac{2(2n-1)c}{qn} \int \frac{dx}{X^n} \\ \text{or} \\ \frac{(2n)!}{(n!)^2} \left(\frac{c}{q} \right)^n \left[\frac{2cx+b}{q} \sum_{r=1}^n \left(\frac{q}{cX} \right)^r \left(\frac{(r-1)!r!}{(2r)!} \right) + \int \frac{dx}{X} \right] \end{cases}$$

$$114. \int \frac{x dx}{X} = \frac{1}{2c} \log X - \frac{b}{2c} \int \frac{dx}{X}$$

$$115. \int \frac{x dx}{X^2} = \frac{bx+2a}{qX} - \frac{b}{q} \int \frac{dx}{X}$$

$$116. \int \frac{x dx}{X^{n+1}} = -\frac{2a+bx}{nqX^n} - \frac{b(2n-1)}{nq} \int \frac{dx}{X^n}$$

$$117. \int \frac{x^2}{X} dx = \frac{x}{c} - \frac{b}{2c^2} \log X + \frac{b^2-2ac}{2c^2} \int \frac{dx}{X}$$

$$118. \int \frac{x^2}{X^2} dx = \frac{(b^2-2ac)x+ab}{cqX} + \frac{2a}{q} \int \frac{dx}{X}$$

$$119. \int \frac{x^m dx}{X^{n+1}} = -\frac{x^{m-1}}{(2n-m+1)cX^n} - \frac{n-m+1}{2n-m+1} \cdot \frac{b}{c} \int \frac{x^{m-1} dx}{X^{n+1}} \\ + \frac{m-1}{2n-m+1} \cdot \frac{a}{c} \int \frac{x^{m-2} dx}{X^{n+1}}$$

$$120. \int \frac{dx}{xX} = \frac{1}{2a} \log \frac{x^2}{X} - \frac{b}{2a} \int \frac{dx}{X}$$

$$121. \int \frac{dx}{x^2 X} = \frac{b}{2a^2} \log \frac{X}{x^2} - \frac{1}{ax} + \left(\frac{b^2}{2a^2} - \frac{c}{a} \right) \int \frac{dx}{X}$$

$$122. \int \frac{dx}{xX^n} = \frac{1}{2a(n-1)X^{n-1}} - \frac{b}{2a} \int \frac{dx}{X^n} + \frac{1}{a} \int \frac{dx}{xX^{n-1}}$$

$$123. \int \frac{dx}{x^m X^{n+1}} = -\frac{1}{(m-1)ax^{m-1}X^n} - \frac{n+m-1}{m-1} \cdot \frac{b}{a} \int \frac{dx}{x^{m-1}X^{n+1}} \\ - \frac{2n+m-1}{m-1} \cdot \frac{c}{a} \int \frac{dx}{x^{m-2}X^{n+1}}$$

FORMS CONTAINING $\sqrt{a+bx}$

$$124. \int \sqrt{a+bx} dx = \frac{2}{3b} \sqrt{(a+bx)^3}$$

$$125. \int x\sqrt{a+bx} dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2}$$

$$126. \int x^2 \sqrt{a+bx} dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3}$$

$$127. \int x^m \sqrt{a+bx} dx = \begin{cases} \frac{2}{b(2m+3)} \left[x^m \sqrt{(a+bx)^3} - ma \int x^{m-1} \sqrt{a+bx} dx \right] \\ \text{or} \\ \frac{2}{b^{m+1}} \sqrt{a+bx} \sum_{r=0}^m \frac{m!(-a)^{m-r}}{r!(m-r)!(2r+3)} (a+bx)^{r+1} \end{cases}$$

INTEGRALS (Continued)

128. $\int \frac{\sqrt{a+bx}}{x} dx = 2\sqrt{a+bx} + a \int \frac{dx}{x\sqrt{a+bx}}$
129. $\int \frac{\sqrt{a+bx}}{x^2} dx = -\frac{\sqrt{a+bx}}{x} + \frac{b}{2} \int \frac{dx}{x\sqrt{a+bx}}$
130. $\int \frac{\sqrt{a+bx}}{x^m} dx = -\frac{1}{(m-1)a} \left[\frac{\sqrt{(a+bx)^3}}{x^{m-1}} + \frac{(2m-5)b}{2} \int \frac{\sqrt{a+bx}}{x^{m-1}} dx \right]$
131. $\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}$
132. $\int \frac{x dx}{\sqrt{a+bx}} = -\frac{2(2a-bx)}{3b^2} \sqrt{a+bx}$
133. $\int \frac{x^2 dx}{\sqrt{a+bx}} = \frac{2(8a^2 - 4abx - 3b^2x^2)}{15b^3} \sqrt{a+bx}$
134. $\int \frac{x^m dx}{\sqrt{a+bx}} = \begin{cases} \frac{2}{(2m+1)b} \left[x^m \sqrt{a+bx} - ma \int \frac{x^{m-1} dx}{\sqrt{a+bx}} \right] \\ \text{or} \\ \frac{2(-a)^m \sqrt{a+bx}}{b^{m+1}} \sum_{r=0}^m \frac{(-1)^r m! (a+bx)^r}{(2r+1)r!(m-r)! a^r} \end{cases}$
135. $\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}} \log \left(\frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right), \quad (a > 0)$
136. $\int \frac{dx}{x\sqrt{a+bx}} = \frac{2}{\sqrt{-a}} \tan^{-1} \sqrt{\frac{a+bx}{-a}}, \quad (a < 0)$
137. $\int \frac{dx}{x^2\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{ax} - \frac{b}{2a} \int \frac{dx}{x\sqrt{a+bx}}$
138. $\int \frac{dx}{x^n \sqrt{a+bx}} = \begin{cases} -\frac{\sqrt{a+bx}}{(n-1)ax^{n-1}} - \frac{(2n-3)b}{(2n-2)a} \int \frac{dx}{x^{n-1}\sqrt{a+bx}} \\ \frac{(2n-2)!}{[(n-1)!]^2} \left[-\frac{\sqrt{a+bx}}{a} \sum_{r=1}^{n-1} \frac{r!(r-1)!}{x^r 2(r)!} \left(-\frac{b}{4a} \right)^{n-r-1} \right. \\ \left. + \left(-\frac{b}{4a} \right)^{n-1} \int \frac{dx}{x\sqrt{a+bx}} \right] \end{cases}$
139. $\int (a+bx)^{\pm \frac{n}{2}} dx = \frac{2(a+bx)^{\frac{2\pm n}{2}}}{b(2\pm n)}$
140. $\int x(a+bx)^{\pm \frac{n}{2}} dx = \frac{2}{b^2} \left[\frac{(a+bx)^{\frac{4\pm n}{2}}}{4\pm n} - \frac{a(a+bx)^{\frac{2\pm n}{2}}}{2\pm n} \right]$
141. $\int \frac{dx}{x(a+bx)^{\frac{m}{2}}} = \frac{1}{a} \int \frac{dx}{x(a+bx)^{\frac{m-2}{2}}} - \frac{b}{a} \int \frac{dx}{(a+bx)^{\frac{m}{2}}}$

INTEGRALS (Continued)

$$142. \int \frac{(a+bx)^{n/2} dx}{x} = b \int (a+bx)^{(n-2)/2} dx + a \int \frac{(a+bx)^{(n-2)/2}}{x} dx$$

$$143. \int f(x, \sqrt{a+bx}) dx = \frac{2}{b} \int f\left(\frac{z^2-a}{b}, z\right) z dz, \quad (z = \sqrt{a+bx})$$

FORMS CONTAINING $\sqrt{a+bx}$ and $\sqrt{c+dx}$

$$u = a+bx \quad v = c+dx \quad k = ad-bc$$

If $k=0$, then, $v = \frac{c}{a}u$, and formulas starting with 124 should be used in place of these.

$$144. \int \frac{dx}{\sqrt{uv}} = \begin{cases} \frac{2}{\sqrt{bd}} \tanh^{-1} \frac{\sqrt{bd}uv}{bv}, & bd > 0, k < 0 \\ \text{or} \\ \frac{2}{\sqrt{bd}} \tanh^{-1} \frac{\sqrt{bd}uv}{du}, & bd > 0, k > 0. \\ \text{or} \\ \frac{1}{\sqrt{bd}} \log \frac{(bv + \sqrt{bd}uv)^2}{v}, & (bd > 0) \end{cases}$$

$$145. \int \frac{dx}{\sqrt{uv}} = \begin{cases} \frac{2}{\sqrt{-bd}} \tan^{-1} \frac{\sqrt{-bd}uv}{bv} \\ \text{or} \\ -\frac{1}{\sqrt{-bd}} \sin^{-1} \left(\frac{2bdx + ad + bc}{|k|} \right), & (bd < 0) \end{cases}$$

$$146. \int \sqrt{uv} dx = \frac{k+2bv}{4bd} \sqrt{uv} - \frac{k^2}{8bd} \int \frac{dx}{\sqrt{uv}}$$

$$147. \int \frac{dx}{v\sqrt{u}} = \begin{cases} \frac{1}{\sqrt{kd}} \log \frac{d\sqrt{u} - \sqrt{kd}}{d\sqrt{u} + \sqrt{kd}} \\ \text{or} \\ \frac{1}{\sqrt{kd}} \log \frac{(d\sqrt{u} - \sqrt{kd})^2}{v}, & (kd > 0) \end{cases}$$

$$148. \int \frac{dx}{v\sqrt{u}} = \frac{2}{\sqrt{-kd}} \tan^{-1} \frac{d\sqrt{u}}{\sqrt{-kd}}, \quad (kd < 0)$$

$$149. \int \frac{x dx}{\sqrt{uv}} = \frac{\sqrt{uv}}{bd} - \frac{ad+bc}{2bd} \int \frac{dx}{\sqrt{uv}}$$

$$150. \int \frac{dx}{v\sqrt{uv}} = \frac{-2\sqrt{uv}}{kv}$$

$$151. \int \frac{v dx}{\sqrt{uv}} = \frac{\sqrt{uv}}{b} - \frac{k}{2b} \int \frac{dx}{\sqrt{uv}}$$

$$152. \int \frac{\sqrt{v}}{\sqrt{u}} dx = \frac{v}{|v|} \int \frac{v dx}{\sqrt{uv}}$$

$$153. \int v^m \sqrt{u} dx = \frac{1}{(2m+3)d} \left(2v^{m+1} \sqrt{u} + k \int \frac{v^m dx}{\sqrt{u}} \right)$$

INTEGRALS (Continued)

$$154. \int \frac{dx}{v^m \sqrt{u}} = -\frac{1}{(m-1)k} \left(\frac{\sqrt{u}}{v^{m-1}} + \left(m - \frac{3}{2}\right)b \int \frac{dx}{v^{m-1} \sqrt{u}} \right)$$

$$155. \int \frac{v^m dx}{\sqrt{u}} = \begin{cases} \frac{2}{b(2m+1)} \left[v^m \sqrt{u} - mk \int \frac{v^{m-1}}{\sqrt{u}} dx \right] \\ \text{or} \\ \frac{2(m!)^2 \sqrt{u}}{b(2m+1)!} \sum_{r=0}^m \left(-\frac{4k}{b} \right)^{m-r} \frac{(2r)!}{(r!)^2} v^r \end{cases}$$

FORMS CONTAINING $\sqrt{x^2 \pm a^2}$

$$156. \int \sqrt{x^2 \pm a^2} dx = \frac{1}{2} [x\sqrt{x^2 \pm a^2} \pm a^2 \log(x + \sqrt{x^2 \pm a^2})]$$

$$157. \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2})$$

$$158. \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|} \sec^{-1} \frac{x}{a}$$

$$159. \int \frac{dx}{x\sqrt{x^2 + a^2}} = -\frac{1}{a} \log \left(\frac{a + \sqrt{x^2 + a^2}}{x} \right)$$

$$160. \int \frac{\sqrt{x^2 + a^2}}{x} dx = \sqrt{x^2 + a^2} - a \log \left(\frac{a + \sqrt{x^2 + a^2}}{x} \right)$$

$$161. \int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - |a| \sec^{-1} \frac{x}{a}$$

$$162. \int \frac{x dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}$$

$$163. \int x\sqrt{x^2 \pm a^2} dx = \frac{1}{3} \sqrt{(x^2 \pm a^2)^3}$$

$$164. \int \sqrt{(x^2 \pm a^2)^3} dx = \frac{1}{4} \left[x\sqrt{(x^2 \pm a^2)^3} \pm \frac{3a^2 x}{2} \sqrt{x^2 \pm a^2} + \frac{3a^4}{2} \log(x + \sqrt{x^2 \pm a^2}) \right]$$

$$165. \int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2 \sqrt{x^2 \pm a^2}}$$

$$166. \int \frac{x dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-1}{\sqrt{x^2 \pm a^2}}$$

$$167. \int x\sqrt{(x^2 \pm a^2)^3} dx = \frac{1}{5} \sqrt{(x^2 \pm a^2)^5}$$

$$168. \int x^2 \sqrt{x^2 \pm a^2} dx = \frac{x}{4} \sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8} x \sqrt{x^2 \pm a^2} - \frac{a^4}{8} \log(x + \sqrt{x^2 \pm a^2})$$

$$169. \int x^3 \sqrt{x^2 + a^2} dx = \left(\frac{1}{5} x^2 - \frac{2}{15} a^2 \right) \sqrt{(a^2 + x^2)^3}$$

$$170. \int x^3 \sqrt{x^2 - a^2} dx = \frac{1}{5} \sqrt{(x^2 - a^2)^5} + \frac{a^2}{3} \sqrt{(x^2 - a^2)^3}$$

$$171. \int \frac{x^2 dx}{\sqrt{x^2 \pm a^2}} = \frac{x}{2} \sqrt{x^2 \pm a^2} \mp \frac{a^2}{2} \log(x + \sqrt{x^2 \pm a^2})$$

INTEGRALS (Continued)

172. $\int \frac{x^3 dx}{\sqrt{x^2 \pm a^2}} = \frac{1}{3} \sqrt{(x^2 \pm a^2)^3} \mp a^2 \sqrt{x^2 \pm a^2}$
173. $\int \frac{dx}{x^2 \sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{a^2 x}$
174. $\int \frac{dx}{x^3 \sqrt{x^2 + a^2}} = \frac{\sqrt{x^2 + a^2}}{2a^2 x^2} + \frac{1}{2a^3} \log \frac{a + \sqrt{x^2 + a^2}}{x}$
175. $\int \frac{dx}{x^3 \sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{2a^2 x^2} + \frac{1}{2|a^3|} \sec^{-1} \frac{x}{a}$
176. $\int x^2 \sqrt{(x^2 \pm a^2)^3} dx = \frac{x}{6} \sqrt{(x^2 \pm a^2)^5} \mp \frac{a^2 x}{24} \sqrt{(x^2 \pm a^2)^3} - \frac{a^4 x}{16} \sqrt{x^2 \pm a^2}$
 $\mp \frac{a^6}{16} \log (x + \sqrt{x^2 \pm a^2})$
177. $\int x^3 \sqrt{(x^2 \pm a^2)^3} dx = \frac{1}{7} \sqrt{(x^2 \pm a^2)^7} \mp \frac{a^2}{5} \sqrt{(x^2 \pm a^2)^5}$
178. $\int \frac{\sqrt{x^2 \pm a^2} dx}{x^2} = -\frac{\sqrt{x^2 \pm a^2}}{x} + \log (x + \sqrt{x^2 \pm a^2})$
179. $\int \frac{\sqrt{x^2 + a^2}}{x^3} dx = -\frac{\sqrt{x^2 + a^2}}{2x^2} - \frac{1}{2a} \log \frac{a + \sqrt{x^2 + a^2}}{x}$
180. $\int \frac{\sqrt{x^2 - a^2}}{x^3} dx = -\frac{\sqrt{x^2 - a^2}}{2x^2} + \frac{1}{2|a|} \sec^{-1} \frac{x}{a}$
181. $\int \frac{\sqrt{x^2 \pm a^2}}{x^4} dx = \mp \frac{\sqrt{(x^2 \pm a^2)^3}}{3a^2 x^3}$
182. $\int \frac{x^2 dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-x}{\sqrt{x^2 \pm a^2}} + \log (x + \sqrt{x^2 \pm a^2})$
183. $\int \frac{x^3 dx}{\sqrt{(x^2 \pm a^2)^3}} = \sqrt{x^2 \pm a^2} \pm \frac{a^2}{\sqrt{x^2 \pm a^2}}$
184. $\int \frac{dx}{x \sqrt{(x^2 + a^2)^3}} = \frac{1}{a^2 \sqrt{x^2 + a^2}} - \frac{1}{a^3} \log \frac{a + \sqrt{x^2 + a^2}}{x}$
185. $\int \frac{dx}{x \sqrt{(x^2 - a^2)^3}} = -\frac{1}{a^2 \sqrt{x^2 - a^2}} - \frac{1}{|a^3|} \sec^{-1} \frac{x}{a}$
186. $\int \frac{dx}{x^2 \sqrt{(x^2 \pm a^2)^3}} = -\frac{1}{a^4} \left[\frac{\sqrt{x^2 \pm a^2}}{x} + \frac{x}{\sqrt{x^2 \pm a^2}} \right]$
187. $\int \frac{dx}{x^3 \sqrt{(x^2 + a^2)^3}} = -\frac{1}{2a^2 x^2 \sqrt{x^2 + a^2}} - \frac{3}{2a^4 \sqrt{x^2 + a^2}} + \frac{3}{2a^5} \log \frac{a + \sqrt{x^2 + a^2}}{x}$
188. $\int \frac{dx}{x^3 \sqrt{(x^2 - a^2)^3}} = \frac{1}{2a^2 x^2 \sqrt{x^2 - a^2}} - \frac{3}{2a^4 \sqrt{x^2 - a^2}} - \frac{3}{2|a^5|} \sec^{-1} \frac{x}{a}$
189. $\int \frac{x^m}{\sqrt{x^2 \pm a^2}} dx = \frac{1}{m} x^{m-1} \sqrt{x^2 \pm a^2} \mp \frac{m-1}{m} a^2 \int \frac{x^{m-2}}{\sqrt{x^2 \pm a^2}} dx$

INTEGRALS (Continued)

$$190. \int \frac{x^{2m}}{\sqrt{x^2 \pm a^2}} dx = \frac{(2m)!}{2^{2m}(m!)^2} \left[\sqrt{x^2 \pm a^2} \sum_{r=1}^m \frac{r!(r-1)!}{(2r)!} (\mp a^2)^{m-r} (2x)^{2r-1} + (\mp a^2)^m \log(x + \sqrt{x^2 \pm a^2}) \right]$$

$$191. \int \frac{x^{2m+1}}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2} \sum_{r=0}^m \frac{(2r)!(m!)^2}{(2m+1)!(r!)^2} (\mp 4a^2)^{m-r} x^{2r}$$

$$192. \int \frac{dx}{x^m \sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{(m-1)a^2 x^{m-1}} \mp \frac{(m-2)}{(m-1)a^2} \int \frac{dx}{x^{m-2} \sqrt{x^2 \pm a^2}}$$

$$193. \int \frac{dx}{x^{2m} \sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2} \sum_{r=0}^{m-1} \frac{(m-1)!m!(2r)!2^{2m-2r-1}}{(r!)^2(2m)!(\mp a^2)^{m-r} x^{2r+1}}$$

$$194. \int \frac{dx}{x^{2m+1} \sqrt{x^2 + a^2}} = \frac{(2m)!}{(m!)^2} \left[\frac{\sqrt{x^2 + a^2}}{a^2} \sum_{r=1}^m (-1)^{m-r+1} \frac{r!(r-1)!}{2(2r)!(4a^2)^{m-r} x^{2r}} + \frac{(-1)^{m+1}}{2^{2m} a^{2m+1}} \log \frac{\sqrt{x^2 + a^2} + a}{x} \right]$$

$$195. \int \frac{dx}{x^{2m+1} \sqrt{x^2 - a^2}} = \frac{(2m)!}{(m!)^2} \left[\frac{\sqrt{x^2 - a^2}}{a^2} \sum_{r=1}^m \frac{r!(r-1)!}{2(2r)!(4a^2)^{m-r} x^{2r}} + \frac{1}{2^{2m}|a|^{2m+1}} \sec^{-1} \frac{x}{a} \right]$$

$$196. \int \frac{dx}{(x-a)\sqrt{x^2 - a^2}} = -\frac{\sqrt{x^2 - a^2}}{a(x-a)}$$

$$197. \int \frac{dx}{(x+a)\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a(x+a)}$$

$$198. \int f(x, \sqrt{x^2 + a^2}) dx = a \int f(a \tan u, a \sec u) \sec^2 u du, \quad \left(u = \tan^{-1} \frac{x}{a}, a > 0\right)$$

$$199. \int f(x, \sqrt{x^2 - a^2}) dx = a \int f(a \sec u, a \tan u) \sec u \tan u du, \quad \left(u = \sec^{-1} \frac{x}{a}, a > 0\right)$$

FORMS CONTAINING $\sqrt{a^2 - x^2}$

$$200. \int \sqrt{a^2 - x^2} dx = \frac{1}{2} \left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{|a|} \right]$$

$$201. \int \frac{dx}{\sqrt{a^2 - x^2}} = \begin{cases} \sin^{-1} \frac{x}{|a|} \\ \text{or} \\ -\cos^{-1} \frac{x}{|a|} \end{cases}$$

$$202. \int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a} \log \left(\frac{a + \sqrt{a^2 - x^2}}{x} \right)$$

$$203. \int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \log \left(\frac{a + \sqrt{a^2 - x^2}}{x} \right)$$

$$204. \int \frac{x dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2}$$

$$205. \int x\sqrt{a^2 - x^2} dx = -\frac{1}{3} \sqrt{(a^2 - x^2)^3}$$

INTEGRALS (Continued)

206. $\int \sqrt{(a^2 - x^2)^3} dx = \frac{1}{4} \left[x\sqrt{(a^2 - x^2)^3} + \frac{3a^2x}{2}\sqrt{a^2 - x^2} + \frac{3a^4}{2}\sin^{-1} \frac{x}{|a|} \right]$
207. $\int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2\sqrt{a^2 - x^2}}$
208. $\int \frac{x dx}{\sqrt{(a^2 - x^2)^3}} = \frac{1}{\sqrt{a^2 - x^2}}$
209. $\int x\sqrt{(a^2 - x^2)^3} dx = -\frac{1}{5}\sqrt{(a^2 - x^2)^5}$
210. $\int x^2\sqrt{a^2 - x^2} dx = -\frac{x}{4}\sqrt{(a^2 - x^2)^3} + \frac{a^2}{8} \left(x\sqrt{a^2 - x^2} + a^2\sin^{-1} \frac{x}{|a|} \right)$
211. $\int x^3\sqrt{a^2 - x^2} dx = \left(-\frac{1}{5}x^2 - \frac{2}{15}a^2\right)\sqrt{(a^2 - x^2)^3}$
212. $\int x^2\sqrt{(a^2 - x^2)^3} dx = -\frac{1}{6}x\sqrt{(a^2 - x^2)^5} + \frac{a^2x}{24}\sqrt{(a^2 - x^2)^3} + \frac{a^4x}{16}\sqrt{a^2 - x^2} + \frac{a^6}{16}\sin^{-1} \frac{x}{|a|}$
213. $\int x^3\sqrt{(a^2 - x^2)^3} dx = \frac{1}{7}\sqrt{(a^2 - x^2)^7} - \frac{a^2}{5}\sqrt{(a^2 - x^2)^5}$
214. $\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1} \frac{x}{|a|}$
215. $\int \frac{dx}{x^2\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2x}$
216. $\int \frac{\sqrt{a^2 - x^2}}{x^2} dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1} \frac{x}{|a|}$
217. $\int \frac{\sqrt{a^2 - x^2}}{x^3} dx = -\frac{\sqrt{a^2 - x^2}}{2x^2} + \frac{1}{2a}\log \frac{a + \sqrt{a^2 - x^2}}{x}$
218. $\int \frac{\sqrt{a^2 - x^2}}{x^4} dx = -\frac{\sqrt{(a^2 - x^2)^3}}{3a^2x^3}$
219. $\int \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1} \frac{x}{|a|}$
220. $\int \frac{x^3 dx}{\sqrt{a^2 - x^2}} = -\frac{2}{3}(a^2 - x^2)^{3/2} - x^2(a^2 - x^2)^{1/2} = -\frac{1}{3}\sqrt{a^2 - x^2}(x^2 + 2a^2)$
221. $\int \frac{x^3 dx}{\sqrt{(a^2 - x^2)^3}} = 2(a^2 - x^2)^{1/2} + \frac{x^2}{(a^2 - x^2)^{1/2}} = -\frac{a^2}{\sqrt{a^2 - x^2}} + \sqrt{a^2 - x^2}$
222. $\int \frac{dx}{x^3\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{2a^2x^2} - \frac{1}{2a^3}\log \frac{a + \sqrt{a^2 - x^2}}{x}$
223. $\int \frac{dx}{x\sqrt{(a^2 - x^2)^3}} = \frac{1}{a^2\sqrt{a^2 - x^2}} - \frac{1}{a^3}\log \frac{a + \sqrt{a^2 - x^2}}{x}$
224. $\int \frac{dx}{x^2\sqrt{(a^2 - x^2)^3}} = \frac{1}{a^4} \left[-\frac{\sqrt{a^2 - x^2}}{x} + \frac{x}{\sqrt{a^2 - x^2}} \right]$
225. $\int \frac{dx}{x^3\sqrt{(a^2 - x^2)^3}} = -\frac{1}{2a^2x^2\sqrt{a^2 - x^2}} + \frac{3}{2a^4\sqrt{a^2 - x^2}} - \frac{3}{2a^5}\log \frac{a + \sqrt{a^2 - x^2}}{x}$

INTEGRALS (Continued)

226. $\int \frac{x^m}{\sqrt{a^2 - x^2}} dx = -\frac{x^{m-1}\sqrt{a^2 - x^2}}{m} + \frac{(m-1)a^2}{m} \int \frac{x^{m-2}}{\sqrt{a^2 - x^2}} dx$
227. $\int \frac{x^{2m}}{\sqrt{a^2 - x^2}} dx = \frac{(2m)!}{(m!)^2} \left[-\sqrt{a^2 - x^2} \sum_{r=1}^m \frac{r!(r-1)!}{2^{2m-2r+1}(2r)!} a^{2m-2r} x^{2r-1} + \frac{a^{2m}}{2^{2m}} \sin^{-1} \frac{x}{|a|} \right]$
228. $\int \frac{x^{2m+1}}{\sqrt{a^2 - x^2}} dx = -\sqrt{a^2 - x^2} \sum_{r=0}^m \frac{(2r)!(m!)^2}{(2m+1)!(r!)^2} (4a^2)^{m-r} x^{2r}$
229. $\int \frac{dx}{x^m \sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{(m-1)a^2 x^{m-1}} + \frac{m-2}{(m-1)a^2} \int \frac{dx}{x^{m-2} \sqrt{a^2 - x^2}}$
230. $\int \frac{ax}{x^{2m} \sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2} \sum_{r=0}^{m-1} \frac{(m-1)!m!(2r)!2^{2m-2r-1}}{(r!)^2(2m)!a^{2m-2r}x^{2r+1}}$
231. $\int \frac{dx}{x^{2m+1} \sqrt{a^2 - x^2}} = \frac{(2m)!}{(m!)^2} \left[-\frac{\sqrt{a^2 - x^2}}{a^2} \sum_{r=1}^m \frac{r!(r-1)!}{2(2r)!(4a^2)^{m-r}x^{2r}} + \frac{1}{2^{2m}a^{2m+1}} \log \frac{a - \sqrt{a^2 - x^2}}{x} \right]$
232. $\int \frac{dx}{(b^2 - x^2)\sqrt{a^2 - x^2}} = \frac{1}{2b\sqrt{a^2 - b^2}} \log \frac{(b\sqrt{a^2 - x^2} + x\sqrt{a^2 - b^2})^2}{b^2 - x^2}, \quad (a^2 > b^2)$
233. $\int \frac{dx}{(b^2 - x^2)\sqrt{a^2 - x^2}} = \frac{1}{b\sqrt{b^2 - a^2}} \tan^{-1} \frac{x\sqrt{b^2 - a^2}}{b\sqrt{a^2 - x^2}}, \quad (b^2 > a^2)$
234. $\int \frac{dx}{(b^2 + x^2)\sqrt{a^2 - x^2}} = \frac{1}{b\sqrt{a^2 + b^2}} \tan^{-1} \frac{x\sqrt{a^2 + b^2}}{b\sqrt{a^2 - x^2}}$
235. $\int \frac{\sqrt{a^2 - x^2}}{b^2 + x^2} dx = \frac{\sqrt{a^2 + b^2}}{|b|} \sin^{-1} \frac{x\sqrt{a^2 + b^2}}{|a|\sqrt{x^2 + b^2}} - \sin^{-1} \frac{x}{|a|}$
236. $\int f(x, \sqrt{a^2 - x^2}) dx = a \int f(a \sin u, a \cos u) \cos u du, \quad \left(u = \sin^{-1} \frac{x}{a}, a > 0\right)$

FORMS CONTAINING $\sqrt{a + bx + cx^2}$

$$X = a + bx + cx^2, \quad q = 4ac - b^2, \quad \text{and } k = \frac{4c}{q}$$

If $q = 0$, then $\sqrt{X} = \sqrt{c} \left| x + \frac{b}{2c} \right|$

$$237. \int \frac{dx}{\sqrt{X}} = \begin{cases} \frac{1}{\sqrt{c}} \log(2\sqrt{cX} + 2cx + b) \\ \text{or} \\ \frac{1}{\sqrt{c}} \sinh^{-1} \frac{2cx + b}{\sqrt{q}}, \quad (c > 0) \end{cases}$$

$$238. \int \frac{dx}{\sqrt{X}} = -\frac{1}{\sqrt{-c}} \sin^{-1} \frac{2cx + b}{\sqrt{-q}}, \quad (c < 0)$$

$$239. \int \frac{dx}{X\sqrt{X}} = \frac{2(2cx + b)}{q\sqrt{X}}$$

INTEGRALS (Continued)

$$240. \int \frac{dx}{X^2\sqrt{X}} = \frac{2(2cx+b)}{3q\sqrt{X}} \left(\frac{1}{X} + 2k \right)$$

$$241. \int \frac{dx}{X^n\sqrt{X}} = \begin{cases} \frac{2(2cx+b)\sqrt{X}}{(2n-1)qX^n} + \frac{2k(n-1)}{2n-1} \int \frac{dx}{X^{n-1}\sqrt{X}} \\ \text{or} \\ \frac{(2cx+b)(n!)(n-1)!4^n k^{n-1}}{q[(2n)!]\sqrt{X}} \sum_{r=0}^{n-1} \frac{(2r)!}{(4kX)^r (r!)^2} \end{cases}$$

$$242. \int \sqrt{X} dx = \frac{(2cx+b)\sqrt{X}}{4c} + \frac{1}{2k} \int \frac{dx}{\sqrt{X}}$$

$$243. \int X\sqrt{X} dx = \frac{(2cx+b)\sqrt{X}}{8c} \left(X + \frac{3}{2k} \right) + \frac{3}{8k^2} \int \frac{dx}{\sqrt{X}}$$

$$244. \int X^2\sqrt{X} dx = \frac{(2cx+b)\sqrt{X}}{12c} \left(X^2 + \frac{5X}{4k} + \frac{15}{8k^2} \right) + \frac{5}{16k^3} \int \frac{dx}{\sqrt{X}}$$

$$245. \int X^n\sqrt{X} dx = \begin{cases} \frac{(2cx+b)X^n\sqrt{X}}{4(n+1)c} + \frac{2n+1}{2(n+1)k} \int X^{n-1}\sqrt{X} dx \\ \text{or} \\ \frac{(2n+2)!}{[(n+1)!]^2(4k)^{n+1}} \left[\frac{k(2cx+b)\sqrt{X}}{c} \sum_{r=0}^n \frac{r!(r+1)!(4kX)^r}{(2r+2)!} + \int \frac{dx}{\sqrt{X}} \right] \end{cases}$$

$$246. \int \frac{x dx}{\sqrt{X}} = \frac{\sqrt{X}}{c} - \frac{b}{2c} \int \frac{dx}{\sqrt{X}}$$

$$247. \int \frac{x dx}{X\sqrt{X}} = -\frac{2(bx+2a)}{q\sqrt{X}}$$

$$248. \int \frac{x dx}{X^n\sqrt{X}} = -\frac{\sqrt{X}}{(2n-1)cX^n} - \frac{b}{2c} \int \frac{dx}{X^n\sqrt{X}}$$

$$249. \int \frac{x^2 dx}{\sqrt{X}} = \left(\frac{x}{2c} - \frac{3b}{4c^2} \right) \sqrt{X} + \frac{3b^2 - 4ac}{8c^2} \int \frac{dx}{\sqrt{X}}$$

$$250. \int \frac{x^2 dx}{X\sqrt{X}} = \frac{(2b^2 - 4ac)x + 2ab}{cq\sqrt{X}} + \frac{1}{c} \int \frac{dx}{\sqrt{X}}$$

$$251. \int \frac{x^2 dx}{X^n\sqrt{X}} = \frac{(2b^2 - 4ac)x + 2ab}{(2n-1)cqX^{n-1}\sqrt{X}} + \frac{4ac + (2n-3)b^2}{(2n-1)cq} \int \frac{dx}{X^{n-1}\sqrt{X}}$$

$$252. \int \frac{x^3 dx}{\sqrt{X}} = \left(\frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^2}{8c^3} - \frac{2a}{3c^2} \right) \sqrt{X} + \left(\frac{3ab}{4c^2} - \frac{5b^3}{16c^3} \right) \int \frac{dx}{\sqrt{X}}$$

$$253. \int \frac{x^n dx}{\sqrt{X}} = \frac{1}{nc} x^{n-1} \sqrt{X} - \frac{(2n-1)b}{2nc} \int \frac{x^{n-1} dx}{\sqrt{X}} - \frac{(n-1)a}{nc} \int \frac{x^{n-2} dx}{\sqrt{X}}$$

$$254. \int x\sqrt{X} dx = \frac{X\sqrt{X}}{3c} - \frac{b(2cx+b)}{8c^2} \sqrt{X} - \frac{b}{4ck} \int \frac{dx}{\sqrt{X}}$$

$$255. \int xX\sqrt{X} dx = \frac{X^2\sqrt{X}}{5c} - \frac{b}{2c} \int X\sqrt{X} dx$$

INTEGRALS (Continued)

$$256. \int x X^n \sqrt{X} dx = \frac{X^{n+1} \sqrt{X}}{(2n+3)c} - \frac{b}{2c} \int X^n \sqrt{X} dx$$

$$257. \int x^2 \sqrt{X} dx = \left(x - \frac{5b}{6c}\right) \frac{X \sqrt{X}}{4c} + \frac{5b^2 - 4ac}{16c^2} \int \sqrt{X} dx$$

$$258. \int \frac{dx}{x\sqrt{X}} = -\frac{1}{\sqrt{a}} \log \frac{2\sqrt{aX} + bx + 2a}{x}, \quad (a > 0)$$

$$259. \int \frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}} \sin^{-1} \left(\frac{bx + 2a}{|x|\sqrt{-a}} \right), \quad (a < 0)$$

$$260. \int \frac{dx}{x\sqrt{X}} = -\frac{2\sqrt{X}}{bx}, \quad (a = 0)$$

$$261. \int \frac{dx}{x^2 \sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2a} \int \frac{dx}{x\sqrt{X}}$$

$$262. \int \frac{\sqrt{X} dx}{x} = \sqrt{X} + \frac{b}{2} \int \frac{dx}{\sqrt{X}} + a \int \frac{dx}{x\sqrt{X}}$$

$$263. \int \frac{\sqrt{X} dx}{x^2} = -\frac{\sqrt{X}}{x} + \frac{b}{2} \int \frac{dx}{x\sqrt{X}} + c \int \frac{dx}{\sqrt{X}}$$

FORMS INVOLVING $\sqrt{2ax - x^2}$

$$264. \int \sqrt{2ax - x^2} dx = \frac{1}{2} \left[(x-a)\sqrt{2ax - x^2} + a^2 \sin^{-1} \frac{x-a}{|a|} \right]$$

$$265. \int \frac{dx}{\sqrt{2ax - x^2}} = \begin{cases} \cos^{-1} \frac{a-x}{|a|} \\ \text{or} \\ \sin^{-1} \frac{x-a}{|a|} \end{cases}$$

$$266. \int x^n \sqrt{2ax - x^2} dx = \begin{cases} -\frac{x^{n-1}(2ax - x^2)^{3/2}}{n+2} + \frac{(2n+1)a}{n+2} \int x^{n-1} \sqrt{2ax - x^2} dx \\ \text{or} \\ \sqrt{2ax - x^2} \left[\frac{x^{n+1}}{n+2} - \sum_{r=0}^n \frac{(2n+1)!(r!)^2 a^{n-r+1}}{2^{n-r}(2r+1)!(n+2)!n!} x^r \right] \\ + \frac{(2n+1)!a^{n+2}}{2^n n!(n+2)!} \sin^{-1} \frac{x-a}{|a|} \end{cases}$$

$$267. \int \frac{\sqrt{2ax - x^2}}{x^n} dx = \frac{(2ax - x^2)^{3/2}}{(3-2n)ax^n} + \frac{n-3}{(2n-3)a} \int \frac{\sqrt{2ax - x^2}}{x^{n-1}} dx$$

$$268. \int \frac{x^n dx}{\sqrt{2ax - x^2}} = \begin{cases} -\frac{x^{n-1} \sqrt{2ax - x^2}}{n} + \frac{a(2n-1)}{n} \int \frac{x^{n-1}}{\sqrt{2ax - x^2}} dx \\ \text{or} \\ -\sqrt{2ax - x^2} \sum_{r=1}^n \frac{(2n)!(r-1)!a^{n-r}}{2^{n-r}(2r)!(n!)^2} x^{r-1} + \frac{(2n)!a^n}{2^n(n!)^2} \sin^{-1} \frac{x-a}{|a|} \end{cases}$$

INTEGRALS (Continued)

$$269. \int \frac{dx}{x^n \sqrt{2ax - x^2}} = \begin{cases} \frac{\sqrt{2ax - x^2}}{a(1-2n)x^n} + \frac{n-1}{(2n-1)a} \int \frac{dx}{x^{n-1} \sqrt{2ax - x^2}} \\ \text{or} \\ -\sqrt{2ax - x^2} \sum_{r=0}^{n-1} \frac{2^{n-r}(n-1)!n!(2r)!}{(2n)!(r!)^2 a^{n-r} x^{r+1}} \end{cases}$$

$$270. \int \frac{dx}{(2ax - x^2)^{3/2}} = \frac{x-a}{a^2 \sqrt{2ax - x^2}}$$

$$271. \int \frac{x dx}{(2ax - x^2)^{3/2}} = \frac{x}{a \sqrt{2ax - x^2}}$$

MISCELLANEOUS ALGEBRAIC FORMS

$$272. \int \frac{dx}{\sqrt{2ax + x^2}} = \log(x + a + \sqrt{2ax + x^2})$$

$$273. \int \sqrt{ax^2 + c} dx = \frac{x}{2} \sqrt{ax^2 + c} + \frac{c}{2\sqrt{a}} \log(x\sqrt{a} + \sqrt{ax^2 + c}), \quad (a > 0)$$

$$274. \int \sqrt{ax^2 + c} dx = \frac{x}{2} \sqrt{ax^2 + c} + \frac{c}{2\sqrt{-a}} \sin^{-1}\left(x\sqrt{-\frac{a}{c}}\right), \quad (a < 0)$$

$$275. \int \sqrt{\frac{1+x}{1-x}} dx = \sin^{-1} x - \sqrt{1-x^2}$$

$$276. \int \frac{dx}{x\sqrt{ax^n + c}} = \begin{cases} \frac{1}{n\sqrt{c}} \log \frac{\sqrt{ax^n + c} - \sqrt{c}}{\sqrt{ax^n + c} + \sqrt{c}} \\ \text{or} \\ \frac{2}{n\sqrt{c}} \log \frac{\sqrt{ax^n + c} - \sqrt{c}}{\sqrt{x^n}}, \quad (c > 0) \end{cases}$$

$$277. \int \frac{dx}{x\sqrt{ax^n + c}} = \frac{2}{n\sqrt{-c}} \sec^{-1} \sqrt{-\frac{ax^n}{c}}, \quad (c < 0)$$

$$278. \int \frac{dx}{\sqrt{ax^2 + c}} = \frac{1}{\sqrt{a}} \log(x\sqrt{a} + \sqrt{ax^2 + c}), \quad (a > 0)$$

$$279. \int \frac{dx}{\sqrt{ax^2 + c}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(x\sqrt{-\frac{a}{c}}\right), \quad (a < 0)$$

$$280. \int (ax^2 + c)^{m+1/2} dx = \begin{cases} \frac{x(ax^2 + c)^{m+1/2}}{2(m+1)} + \frac{(2m+1)c}{2(m+1)} \int (ax^2 + c)^{m-1/2} dx \\ \text{or} \\ x\sqrt{ax^2 + c} \sum_{r=0}^m \frac{(2m+1)!(r!)^2 c^{m-r}}{2^{2m-2r+1} m! (m+1)! (2r+1)!} (ax^2 + c)^r \\ + \frac{(2m+1)! c^{m+1}}{2^{2m+1} m! (m+1)!} \int \frac{dx}{\sqrt{ax^2 + c}} \end{cases}$$

$$281. \int x(ax^2 + c)^{m+1/2} dx = \frac{(ax^2 + c)^{m+3/2}}{(2m+3)a}$$

INTEGRALS (Continued)

$$\begin{aligned}
 282. \int \frac{(ax^2 + c)^{m+1/2}}{x} dx &= \begin{cases} \frac{(ax^2 + c)^{m+1/2}}{2m+1} + c \int \frac{(ax^2 + c)^{m-1/2}}{x} dx \\ \text{or} \\ \sqrt{ax^2 + c} \sum_{r=0}^m \frac{c^{m-r}(ax^2 + c)^r}{2r+1} + c^{m+1} \int \frac{dx}{x\sqrt{ax^2 + c}} \end{cases} \\
 283. \int \frac{dx}{(ax^2 + c)^{m+1/2}} &= \begin{cases} \frac{x}{(2m-1)c(ax^2 + c)^{m-1/2}} + \frac{2m-2}{(2m-1)c} \int \frac{dx}{(ax^2 + c)^{m-1/2}} \\ \text{or} \\ \frac{x}{\sqrt{ax^2 + c}} \sum_{r=0}^{m-1} \frac{2^{2m-2r-1}(m-1)!m!(2r)!}{(2m)!(r!)^2 c^{m-r}(ax^2 + c)^r} \end{cases} \\
 284. \int \frac{dx}{x^m \sqrt{ax^2 + c}} &= -\frac{\sqrt{ax^2 + c}}{(m-1)c x^{m-1}} - \frac{(m-2)a}{(m-1)c} \int \frac{dx}{x^{m-2} \sqrt{ax^2 + c}} \\
 285. \int \frac{1+x^2}{(1-x^2)\sqrt{1+x^4}} dx &= \frac{1}{\sqrt{2}} \log \frac{x\sqrt{2} + \sqrt{1+x^4}}{1-x^2} \\
 286. \int \frac{1-x^2}{(1+x^2)\sqrt{1+x^4}} dx &= \frac{1}{\sqrt{2}} \tan^{-1} \frac{x\sqrt{2}}{\sqrt{1+x^4}} \\
 287. \int \frac{dx}{x\sqrt{x^n + a^2}} &= -\frac{2}{na} \log \frac{a + \sqrt{x^n + a^2}}{\sqrt{x^n}} \\
 288. \int \frac{dx}{x\sqrt{x^n - a^2}} &= -\frac{2}{na} \sin^{-1} \frac{a}{\sqrt{x^n}} \\
 289. \int \sqrt{\frac{x}{a^3 - x^3}} dx &= \frac{2}{3} \sin^{-1} \left(\frac{x}{a} \right)^{3/2}
 \end{aligned}$$

FORMS INVOLVING TRIGONOMETRIC FUNCTIONS

$$\begin{aligned}
 290. \int (\sin ax) dx &= -\frac{1}{a} \cos ax \\
 291. \int (\cos ax) dx &= \frac{1}{a} \sin ax \\
 292. \int (\tan ax) dx &= -\frac{1}{a} \log \cos ax = \frac{1}{a} \log \sec ax \\
 293. \int (\cot ax) dx &= \frac{1}{a} \log \sin ax = -\frac{1}{a} \log \csc ax \\
 294. \int (\sec ax) dx &= \frac{1}{a} \log(\sec ax + \tan ax) = \frac{1}{a} \log \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right) \\
 295. \int (\csc ax) dx &= \frac{1}{a} \log(\csc ax - \cot ax) = \frac{1}{a} \log \tan \frac{ax}{2} \\
 296. \int (\sin^2 ax) dx &= -\frac{1}{2a} \cos ax \sin ax + \frac{1}{2} x = \frac{1}{2} x - \frac{1}{4a} \sin 2ax \\
 297. \int (\sin^3 ax) dx &= -\frac{1}{3a} (\cos ax)(\sin^2 ax + 2) \\
 298. \int (\sin^4 ax) dx &= \frac{3x}{8} - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a} \\
 299. \int (\sin^n ax) dx &= -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n} \int (\sin^{n-2} ax) dx
 \end{aligned}$$

INTEGRALS (Continued)

300. $\int (\sin^{2m} ax) dx = -\frac{\cos ax}{a} \sum_{r=0}^{m-1} \frac{(2m)!(r!)^2}{2^{2m-2r}(2r+1)!(m!)^2} \sin^{2r+1} ax + \frac{(2m)!}{2^{2m}(m!)^2} x$
301. $\int (\sin^{2m+1} ax) dx = -\frac{\cos ax}{a} \sum_{r=0}^m \frac{2^{2m-2r}(m!)^2(2r)!}{(2m+1)!(r!)^2} \sin^{2r} ax$
302. $\int (\cos^2 ax) dx = \frac{1}{2a} \sin ax \cos ax + \frac{1}{2} x = \frac{1}{2} x + \frac{1}{4a} \sin 2ax$
303. $\int (\cos^3 ax) dx = \frac{1}{3a} (\sin ax)(\cos^2 ax + 2)$
304. $\int (\cos^4 ax) dx = \frac{3x}{8} + \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$
305. $\int (\cos^n ax) dx = \frac{1}{na} \cos^{n-1} ax \sin ax + \frac{n-1}{n} \int (\cos^{n-2} ax) dx$
306. $\int (\cos^{2m} ax) dx = \frac{\sin ax}{a} \sum_{r=0}^{m-1} \frac{(2m)!(r!)^2}{2^{2m-2r}(2r+1)!(m!)^2} \cos^{2r+1} ax + \frac{(2m)!}{2^{2m}(m!)^2} x$
307. $\int (\cos^{2m+1} ax) dx = \frac{\sin ax}{a} \sum_{r=0}^m \frac{2^{2m-2r}(m!)^2(2r)!}{(2m+1)!(r!)^2} \cos^{2r} ax$
308. $\int \frac{dx}{\sin^2 ax} = \int (\csc^2 ax) dx = -\frac{1}{a} \cot ax$
309. $\int \frac{dx}{\sin^m ax} = \int (\csc^m ax) dx = -\frac{1}{(m-1)a} \cdot \frac{\cos ax}{\sin^{m-1} ax} + \frac{m-2}{m-1} \int \frac{dx}{\sin^{m-2} ax}$
310. $\int \frac{dx}{\sin^{2m} ax} = \int (\csc^{2m} ax) dx = -\frac{1}{a} \cos ax \sum_{r=0}^{m-1} \frac{2^{2m-2r-1}(m-1)!m!(2r)!}{(2m)!(r!)^2 \sin^{2r+1} ax}$
311. $\int \frac{dx}{\sin^{2m+1} ax} = \int (\csc^{2m+1} ax) dx$
 $= -\frac{1}{a} \cos ax \sum_{r=0}^{m-1} \frac{(2m)!(r!)^2}{2^{2m-2r}(m!)^2(2r+1)! \sin^{2r+2} ax} + \frac{1}{a} \cdot \frac{(2m)!}{2^{2m}(m!)^2} \log \tan \frac{ax}{2}$
312. $\int \frac{dx}{\cos^2 ax} = \int (\sec^2 ax) dx = \frac{1}{a} \tan ax$
313. $\int \frac{dx}{\cos^n ax} = \int (\sec^n ax) dx = \frac{1}{(n-1)a} \cdot \frac{\sin ax}{\cos^{n-1} ax} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} ax}$
314. $\int \frac{dx}{\cos^{2m} ax} = \int (\sec^{2m} ax) dx = \frac{1}{a} \sin ax \sum_{r=0}^{m-1} \frac{2^{2m-2r-1}(m-1)!m!(2r)!}{(2m)!(r!)^2 \cos^{2r+1} ax}$
315. $\int \frac{dx}{\cos^{2m+1} ax} = \int (\sec^{2m+1} ax) dx$
 $= \frac{1}{a} \sin ax \sum_{r=0}^{m-1} \frac{(2m)!(r!)^2}{2^{2m-2r}(m!)^2(2r+1)! \cos^{2r+2} ax} + \frac{1}{a} \cdot \frac{(2m)!}{2^{2m}(m!)^2} \log(\sec ax + \tan ax)$
316. $\int (\sin mx)(\sin nx) dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}, \quad (m^2 \neq n^2)$
317. $\int (\cos mx)(\cos nx) dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}, \quad (m^2 \neq n^2)$
318. $\int (\sin ax)(\cos ax) dx = \frac{1}{2a} \sin^2 ax$

INTEGRALS (Continued)

$$319. \int (\sin mx)(\cos nx) dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}, \quad (m^2 \neq n^2)$$

$$320. \int (\sin^2 ax)(\cos^2 ax) dx = -\frac{1}{32a} \sin 4ax + \frac{x}{8}$$

$$321. \int (\sin ax)(\cos^m ax) dx = -\frac{\cos^{m+1} ax}{(m+1)a}$$

$$322. \int (\sin^m ax)(\cos ax) dx = \frac{\sin^{m+1} ax}{(m+1)a}$$

$$323. \int (\cos^m ax)(\sin^n ax) dx = \begin{cases} \frac{\cos^{m-1} ax \sin^{n+1} ax}{(m+n)a} + \frac{m-1}{m+n} \int (\cos^{m-2} ax)(\sin^n ax) dx \\ \text{or} \\ -\frac{\sin^{n-1} ax \cos^{m+1} ax}{(m+n)a} + \frac{n-1}{m+n} \int (\cos^m ax)(\sin^{n-2} ax) dx \end{cases}$$

$$324. \int \frac{\cos^m ax}{\sin^n ax} dx = \begin{cases} -\frac{\cos^{m+1} ax}{(n-1)a \sin^{n-1} ax} - \frac{m-n+2}{n-1} \int \frac{\cos^m ax}{\sin^{n-2} ax} dx \\ \text{or} \\ \frac{\cos^{m-1} ax}{a(m-n) \sin^{n-1} ax} + \frac{m-1}{m-n} \int \frac{\cos^{m-2} ax}{\sin^n ax} dx \end{cases}$$

$$325. \int \frac{\sin^m ax}{\cos^n ax} dx = \begin{cases} \frac{\sin^{m+1} ax}{a(n-1) \cos^{n-1} ax} - \frac{m-n+2}{n-1} \int \frac{\sin^m ax}{\cos^{n-2} ax} dx \\ \text{or} \\ -\frac{\sin^{m-1} ax}{a(m-n) \cos^{n-1} ax} + \frac{m-1}{m-n} \int \frac{\sin^{m-2} ax}{\cos^n ax} dx \end{cases}$$

$$326. \int \frac{\sin ax}{\cos^2 ax} dx = \frac{1}{a \cos ax} = \frac{\sec ax}{a}$$

$$327. \int \frac{\sin^2 ax}{\cos ax} dx = -\frac{1}{a} \sin ax + \frac{1}{a} \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

$$328. \int \frac{\cos ax}{\sin^2 ax} dx = -\frac{1}{a \sin ax} = -\frac{\csc ax}{a}$$

$$329. \int \frac{dx}{(\sin ax)(\cos ax)} = \frac{1}{a} \log \tan ax$$

$$330. \int \frac{dx}{(\sin ax)(\cos^2 ax)} = \frac{1}{a} \left(\sec ax + \log \tan \frac{ax}{2} \right)$$

$$331. \int \frac{dx}{(\sin ax)(\cos^n ax)} = \frac{1}{a(n-1) \cos^{n-1} ax} + \int \frac{dx}{(\sin ax)(\cos^{n-2} ax)}$$

$$332. \int \frac{dx}{(\sin^2 ax)(\cos ax)} = -\frac{1}{a} \csc ax + \frac{1}{a} \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

$$333. \int \frac{dx}{(\sin^2 ax)(\cos^2 ax)} = -\frac{2}{a} \cot 2ax$$

INTEGRALS (Continued)

$$334. \int \frac{dx}{\sin^m ax \cos^n ax} = \begin{cases} -\frac{1}{a(m-1)(\sin^{m-1} ax)(\cos^{n-1} ax)} \\ \quad + \frac{m+n-2}{m-1} \int \frac{dx}{(\sin^{m-2} ax)(\cos^n ax)} \\ \text{or} \\ \frac{1}{a(n-1)\sin^{m-1} ax \cos^{n-1} ax} - \frac{m+n-2}{n-1} \int \frac{dx}{\sin^m ax \cos^{n-2} ax} \end{cases}$$

$$335. \int \sin(a+bx) dx = -\frac{1}{b} \cos(a+bx)$$

$$336. \int \cos(a+bx) dx = \frac{1}{b} \sin(a+bx)$$

$$337. \int \frac{dx}{1 \pm \sin ax} = \mp \frac{1}{a} \tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right)$$

$$338. \int \frac{dx}{1 + \cos ax} = \frac{1}{a} \tan \frac{ax}{2}$$

$$339. \int \frac{dx}{1 - \cos ax} = -\frac{1}{a} \cot \frac{ax}{2}$$

$$*340. \int \frac{dx}{a+b \sin x} = \begin{cases} \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \frac{a \tan \frac{x}{2} + b}{\sqrt{a^2-b^2}} \\ \text{or} \\ \frac{1}{\sqrt{b^2-a^2}} \log \frac{a \tan \frac{x}{2} + b - \sqrt{b^2-a^2}}{a \tan \frac{x}{2} + b + \sqrt{b^2-a^2}} \end{cases}$$

$$*341. \int \frac{dx}{a+b \cos x} = \begin{cases} \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \frac{\sqrt{a^2-b^2} \tan \frac{x}{2}}{a+b} \\ \text{or} \\ \frac{1}{\sqrt{b^2-a^2}} \log \left(\frac{\sqrt{b^2-a^2} \tan \frac{x}{2} + a + b}{\sqrt{b^2-a^2} \tan \frac{x}{2} - a - b} \right) \end{cases}$$

$$*342. \int \frac{dx}{a+b \sin x + c \cos x} = \begin{cases} \frac{1}{\sqrt{b^2+c^2-a^2}} \log \frac{b - \sqrt{b^2+c^2-a^2} + (a-c) \tan \frac{x}{2}}{b + \sqrt{b^2+c^2-a^2} + (a-c) \tan \frac{x}{2}}, & \text{if } a^2 < b^2 + c^2, a \neq c \\ \text{or} \\ \frac{2}{\sqrt{a^2-b^2-c^2}} \tan^{-1} \frac{b + (a-c) \tan \frac{x}{2}}{\sqrt{a^2-b^2-c^2}}, & \text{if } a^2 > b^2 + c^2 \\ \text{or} \\ \frac{1}{a} \left[\frac{a - (b+c) \cos x - (b-c) \sin x}{a - (b-c) \cos x + (b+c) \sin x} \right], & \text{if } a^2 = b^2 + c^2, a \neq c. \end{cases}$$

*See note 6 on page A-19.

INTEGRALS (Continued)

$$*343. \int \frac{\sin^2 x \, dx}{a + b \cos^2 x} = \frac{1}{b} \sqrt{\frac{a+b}{a}} \tan^{-1} \left(\sqrt{\frac{a}{a+b}} \tan x \right) - \frac{x}{b}, \quad (ab > 0, \text{ or } |a| > |b|)$$

$$*344. \int \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{1}{ab} \tan^{-1} \left(\frac{b \tan x}{a} \right)$$

$$*345. \int \frac{\cos^2 cx}{a^2 + b^2 \sin^2 cx} dx = \frac{\sqrt{a^2 + b^2}}{ab^2 c} \tan^{-1} \frac{\sqrt{a^2 + b^2} \tan cx}{a} - \frac{x}{b^2}$$

$$346. \int \frac{\sin cx \cos cx}{a \cos^2 cx + b \sin^2 cx} dx = \frac{1}{2c(b-a)} \log(a \cos^2 cx + b \sin^2 cx)$$

$$347. \int \frac{\cos cx}{a \cos cx + b \sin cx} dx = \int \frac{dx}{a + b \tan cx} \\ = \frac{1}{c(a^2 + b^2)} [acx + b \log(a \cos cx + b \sin cx)]$$

$$348. \int \frac{\sin cx}{a \sin cx + b \cos cx} dx = \int \frac{dx}{a + b \cot cx} = \frac{1}{c(a^2 + b^2)} [acx - b \log(a \sin cx + b \cos cx)]$$

$$*349. \int \frac{dx}{a \cos^2 x + 2b \cos x \sin x + c \sin^2 x} = \begin{cases} \frac{1}{2\sqrt{b^2 - ac}} \log \frac{c \tan x + b - \sqrt{b^2 - ac}}{c \tan x + b + \sqrt{b^2 - ac}}, & (b^2 > ac) \\ \text{or} \\ \frac{1}{\sqrt{ac - b^2}} \tan^{-1} \frac{c \tan x + b}{\sqrt{ac - b^2}}, & (b^2 < ac) \\ \text{or} \\ -\frac{1}{c \tan x + b}, & (b^2 = ac) \end{cases}$$

$$350. \int \frac{\sin ax}{1 \pm \sin ax} dx = \pm x + \frac{1}{a} \tan \left(\frac{\pi}{4} \mp \frac{ax}{2} \right)$$

$$351. \int \frac{dx}{(\sin ax)(1 \pm \sin ax)} = \frac{1}{a} \tan \left(\frac{\pi}{4} \mp \frac{ax}{2} \right) + \frac{1}{a} \log \tan \frac{ax}{2}$$

$$352. \int \frac{dx}{(1 + \sin ax)^2} = -\frac{1}{2a} \tan \left(\frac{\pi}{4} - \frac{ax}{2} \right) - \frac{1}{6a} \tan^3 \left(\frac{\pi}{4} - \frac{ax}{2} \right)$$

$$353. \int \frac{dx}{(1 - \sin ax)^2} = \frac{1}{2a} \cot \left(\frac{\pi}{4} - \frac{ax}{2} \right) + \frac{1}{6a} \cot^3 \left(\frac{\pi}{4} - \frac{ax}{2} \right)$$

$$354. \int \frac{\sin ax}{(1 + \sin ax)^2} dx = -\frac{1}{2a} \tan \left(\frac{\pi}{4} - \frac{ax}{2} \right) + \frac{1}{6a} \tan^3 \left(\frac{\pi}{4} - \frac{ax}{2} \right)$$

$$355. \int \frac{\sin ax}{(1 - \sin ax)^2} dx = -\frac{1}{2a} \cot \left(\frac{\pi}{4} - \frac{ax}{2} \right) + \frac{1}{6a} \cot^3 \left(\frac{\pi}{4} - \frac{ax}{2} \right)$$

$$356. \int \frac{\sin x \, dx}{a + b \sin x} = \frac{x}{b} - \frac{a}{b} \int \frac{dx}{a + b \sin x}$$

$$357. \int \frac{dx}{(\sin x)(a + b \sin x)} = \frac{1}{a} \log \tan \frac{x}{2} - \frac{b}{a} \int \frac{dx}{a + b \sin x}$$

$$358. \int \frac{dx}{(a + b \sin x)^2} = \frac{b \cos x}{(a^2 - b^2)(a + b \sin x)} + \frac{a}{a^2 - b^2} \int \frac{dx}{a + b \sin x}$$

*See note 6 on page A-19.

INTEGRALS (Continued)

$$359. \int \frac{\sin x \, dx}{(a + b \sin x)^2} = \frac{a \cos x}{(b^2 - a^2)(a + b \sin x)} + \frac{h}{b^2 - a^2} \int \frac{dx}{a + b \sin x}$$

$$*360. \int \frac{dx}{a^2 + b^2 \sin^2 cx} = \frac{1}{ac\sqrt{a^2 + b^2}} \tan^{-1} \frac{\sqrt{a^2 + b^2} \tan cx}{a}$$

$$*361. \int \frac{dx}{a^2 - b^2 \sin^2 cx} = \begin{cases} \frac{1}{ac\sqrt{a^2 - b^2}} \tan^{-1} \frac{\sqrt{a^2 - b^2} \tan cx}{a}, & (a^2 > b^2) \\ \text{or} \\ \frac{1}{2ac\sqrt{b^2 - a^2}} \log \frac{\sqrt{b^2 - a^2} \tan cx + a}{\sqrt{b^2 - a^2} \tan cx - a}, & (a^2 < b^2) \end{cases}$$

$$362. \int \frac{\cos ax}{1 + \cos ax} dx = x - \frac{1}{a} \tan \frac{ax}{2}$$

$$363. \int \frac{\cos ax}{1 - \cos ax} dx = -x - \frac{1}{a} \cot \frac{ax}{2}$$

$$364. \int \frac{dx}{(\cos ax)(1 + \cos ax)} = \frac{1}{a} \log \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right) - \frac{1}{a} \tan \frac{ax}{2}$$

$$365. \int \frac{dx}{(\cos ax)(1 - \cos ax)} = \frac{1}{a} \log \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right) - \frac{1}{a} \cot \frac{ax}{2}$$

$$366. \int \frac{dx}{(1 + \cos ax)^2} = \frac{1}{2a} \tan \frac{ax}{2} + \frac{1}{6a} \tan^3 \frac{ax}{2}$$

$$367. \int \frac{dx}{(1 - \cos ax)^2} = -\frac{1}{2a} \cot \frac{ax}{2} - \frac{1}{6a} \cot^3 \frac{ax}{2}$$

$$368. \int \frac{\cos ax}{(1 + \cos ax)^2} dx = \frac{1}{2a} \tan \frac{ax}{2} - \frac{1}{6a} \tan^3 \frac{ax}{2}$$

$$369. \int \frac{\cos ax}{(1 - \cos ax)^2} dx = \frac{1}{2a} \cot \frac{ax}{2} - \frac{1}{6a} \cot^3 \frac{ax}{2}$$

$$370. \int \frac{\cos x \, dx}{a + b \cos x} = \frac{x}{b} - \frac{a}{b} \int \frac{dx}{a + b \cos x}$$

$$371. \int \frac{dx}{(\cos x)(a + b \cos x)} = \frac{1}{a} \log \tan \left(\frac{x}{2} + \frac{\pi}{4} \right) - \frac{b}{a} \int \frac{dx}{a + b \cos x}$$

$$372. \int \frac{dx}{(a + b \cos x)^2} = \frac{b \sin x}{(b^2 - a^2)(a + b \cos x)} - \frac{a}{b^2 - a^2} \int \frac{dx}{a + b \cos x}$$

$$373. \int \frac{\cos x}{(a + b \cos x)^2} dx = \frac{a \sin x}{(a^2 - b^2)(a + b \cos x)} - \frac{b}{a^2 - b^2} \int \frac{dx}{a + b \cos x}$$

$$*374. \int \frac{dx}{a^2 + b^2 - 2ab \cos cx} = \frac{2}{c(a^2 - b^2)} \tan^{-1} \left(\frac{a + b}{a - b} \tan \frac{cx}{2} \right)$$

$$*375. \int \frac{dx}{a^2 + b^2 \cos^2 cx} = \frac{1}{ac\sqrt{a^2 + b^2}} \tan^{-1} \frac{a \tan cx}{\sqrt{a^2 + b^2}}$$

$$*376. \int \frac{dx}{a^2 - b^2 \cos^2 cx} = \begin{cases} \frac{1}{ac\sqrt{a^2 - b^2}} \tan^{-1} \frac{a \tan cx}{\sqrt{a^2 - b^2}}, & (a^2 > b^2) \\ \text{or} \\ \frac{1}{2ac\sqrt{b^2 - a^2}} \log \frac{a \tan cx - \sqrt{b^2 - a^2}}{a \tan cx + \sqrt{b^2 - a^2}}, & (b^2 > a^2) \end{cases}$$

$$377. \int \frac{\sin ax}{1 \pm \cos ax} dx = \mp \frac{1}{a} \log(1 \pm \cos ax)$$

*See note 6 on page A-19.

INTEGRALS (Continued)

$$378. \int \frac{\cos ax}{1 \pm \sin ax} dx = \pm \frac{1}{a} \log(1 \pm \sin ax)$$

$$379. \int \frac{dx}{(\sin ax)(1 \pm \cos ax)} = \pm \frac{1}{2a(1 \pm \cos ax)} + \frac{1}{2a} \log \tan \frac{ax}{2}$$

$$380. \int \frac{dx}{(\cos ax)(1 \pm \sin ax)} = \mp \frac{1}{2a(1 \pm \sin ax)} + \frac{1}{2a} \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

$$381. \int \frac{\sin ax}{(\cos ax)(1 \pm \cos ax)} dx = \frac{1}{a} \log(\sec ax \pm 1)$$

$$382. \int \frac{\cos ax}{(\sin ax)(1 \pm \sin ax)} dx = -\frac{1}{a} \log(\csc ax \pm 1)$$

$$383. \int \frac{\sin ax}{(\cos ax)(1 \pm \sin ax)} dx = \frac{1}{2a(1 \pm \sin ax)} \pm \frac{1}{2a} \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

$$384. \int \frac{\cos ax}{(\sin ax)(1 \pm \cos ax)} dx = -\frac{1}{2a(1 \pm \cos ax)} \pm \frac{1}{2a} \log \tan \frac{ax}{2}$$

$$385. \int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{a\sqrt{2}} \log \tan\left(\frac{ax}{2} \pm \frac{\pi}{8}\right)$$

$$386. \int \frac{dx}{(\sin ax \pm \cos ax)^2} = \frac{1}{2a} \tan\left(ax \mp \frac{\pi}{4}\right)$$

$$387. \int \frac{dx}{1 + \cos ax \pm \sin ax} = \pm \frac{1}{a} \log\left(1 \pm \tan \frac{ax}{2}\right)$$

$$388. \int \frac{dx}{a^2 \cos^2 cx - b^2 \sin^2 cx} = \frac{1}{2abc} \log \frac{b \tan cx + a}{b \tan cx - a}$$

$$389. \int x(\sin ax) dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax$$

$$390. \int x^2(\sin ax) dx = \frac{2x}{a^2} \sin ax - \frac{a^2 x^2 - 2}{a^3} \cos ax$$

$$391. \int x^3(\sin ax) dx = \frac{3a^2 x^2 - 6}{a^4} \sin ax - \frac{a^2 x^3 - 6x}{a^3} \cos ax$$

$$392. \int x^m \sin ax dx = \begin{cases} -\frac{1}{a} x^m \cos ax + \frac{m}{a} \int x^{m-1} \cos ax dx \\ \text{or} \\ \cos ax \sum_{r=0}^{[m/2]} (-1)^{r+1} \frac{m!}{(m-2r)!} \cdot \frac{x^{m-2r}}{a^{2r+1}} \\ + \sin ax \sum_{r=0}^{[(m-1)/2]} (-1)^r \frac{m!}{(m-2r-1)!} \cdot \frac{x^{m-2r-1}}{a^{2r+2}} \end{cases}$$

Note: $[s]$ means greatest integer $\leq s$; $[3\frac{1}{2}] = 3$, $[\frac{1}{2}] = 0$, etc.

$$393. \int x(\cos ax) dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax$$

$$394. \int x^2(\cos ax) dx = \frac{2x \cos ax}{a^2} + \frac{a^2 x^2 - 2}{a^3} \sin ax$$

$$395. \int x^3(\cos ax) dx = \frac{3a^2 x^2 - 6}{a^4} \cos ax + \frac{a^2 x^3 - 6x}{a^3} \sin ax$$

INTEGRALS (Continued)

$$396. \int x^m (\cos ax) dx = \begin{cases} \frac{x^m \sin ax}{a} - \frac{m}{a} \int x^{m-1} \sin ax dx \\ \text{or} \\ \sin ax \sum_{r=0}^{[m/2]} (-1)^r \frac{m!}{(m-2r)!} \cdot \frac{x^{m-2r}}{a^{2r+1}} \\ + \cos ax \sum_{r=0}^{[(m-1)/2]} (-1)^r \frac{m!}{(m-2r-1)!} \cdot \frac{x^{m-2r-1}}{a^{2r+2}} \end{cases}$$

See note integral 392.

$$397. \int \frac{\sin ax}{x} dx = \sum_{n=0}^r (-1)^n \frac{(ax)^{2n+1}}{(2n+1)(2n+1)!}$$

$$398. \int \frac{\cos ax}{x} dx = \log x + \sum_{n=1}^r (-1)^n \frac{(ax)^{2n}}{2n(2n)!}$$

$$399. \int x(\sin^2 ax) dx = \frac{x^2}{4} - \frac{x \sin 2ax}{4a} - \frac{\cos 2ax}{8a^2}$$

$$400. \int x^2(\sin^2 ax) dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3} \right) \sin 2ax - \frac{x \cos 2ax}{4a^2}$$

$$401. \int x(\sin^3 ax) dx = \frac{x \cos 3ax}{12a} - \frac{\sin 3ax}{36a^2} - \frac{3x \cos ax}{4a} + \frac{3 \sin ax}{4a^2}$$

$$402. \int x(\cos^2 ax) dx = \frac{x^2}{4} + \frac{x \sin 2ax}{4a} + \frac{\cos 2ax}{8a^2}$$

$$403. \int x^2(\cos^2 ax) dx = \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3} \right) \sin 2ax + \frac{x \cos 2ax}{4a^2}$$

$$404. \int x(\cos^3 ax) dx = \frac{x \sin 3ax}{12a} + \frac{\cos 3ax}{36a^2} + \frac{3x \sin ax}{4a} + \frac{3 \cos ax}{4a^2}$$

$$405. \int \frac{\sin ax}{x^m} dx = -\frac{\sin ax}{(m-1)x^{m-1}} + \frac{a}{m-1} \int \frac{\cos ax}{x^{m-1}} dx$$

$$406. \int \frac{\cos ax}{x^m} dx = -\frac{\cos ax}{(m-1)x^{m-1}} - \frac{a}{m-1} \int \frac{\sin ax}{x^{m-1}} dx$$

$$407. \int \frac{x}{1 \pm \sin ax} dx = \mp \frac{x \cos ax}{a(1 \pm \sin ax)} + \frac{1}{a^2} \log(1 \pm \sin ax)$$

$$408. \int \frac{x}{1 + \cos ax} dx = \frac{x}{a} \tan \frac{ax}{2} + \frac{2}{a^2} \log \cos \frac{ax}{2}$$

$$409. \int \frac{x}{1 - \cos ax} dx = -\frac{x}{a} \cot \frac{ax}{2} + \frac{2}{a^2} \log \sin \frac{ax}{2}$$

$$410. \int \frac{x + \sin x}{1 + \cos x} dx = x \tan \frac{x}{2}$$

$$411. \int \frac{x - \sin x}{1 - \cos x} dx = -x \cot \frac{x}{2}$$

$$412. \int \sqrt{1 - \cos ax} dx = -\frac{2 \sin ax}{a\sqrt{1 - \cos ax}} = -\frac{2\sqrt{2}}{a} \cos\left(\frac{ax}{2}\right)$$

$$413. \int \sqrt{1 + \cos ax} dx = \frac{2 \sin ax}{a\sqrt{1 + \cos ax}} = \frac{2\sqrt{2}}{a} \sin\left(\frac{ax}{2}\right)$$

$$414. \int \sqrt{1 + \sin x} dx = \pm 2 \left(\sin \frac{x}{2} - \cos \frac{x}{2} \right),$$

[use + if $(8k-1)\frac{\pi}{2} < x \leq (8k+3)\frac{\pi}{2}$, otherwise -; k an integer]

INTEGRALS (Continued)

415. $\int \sqrt{1 - \sin x} \, dx = \pm 2 \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right),$
 $\left[\text{use} + \text{if } (8k-3)\frac{\pi}{2} < x \leq (8k+1)\frac{\pi}{2}, \text{ otherwise } -; k \text{ an integer} \right]$
416. $\int \frac{dx}{\sqrt{1 - \cos x}} = \pm \sqrt{2} \log \tan \frac{x}{4},$
 $[\text{use} + \text{if } 4k\pi < x < (4k+2)\pi, \text{ otherwise } -; k \text{ an integer}]$
417. $\int \frac{dx}{\sqrt{1 + \cos x}} = \pm \sqrt{2} \log \tan \left(\frac{x + \pi}{4} \right),$
 $[\text{use} + \text{if } (4k-1)\pi < x < (4k+1)\pi, \text{ otherwise } -; k \text{ an integer}]$
418. $\int \frac{dx}{\sqrt{1 - \sin x}} = \pm \sqrt{2} \log \tan \left(\frac{x}{4} - \frac{\pi}{8} \right),$
 $[\text{use} + \text{if } (8k+1)\frac{\pi}{2} < x < (8k+5)\frac{\pi}{2}, \text{ otherwise } -; k \text{ an integer}]$
419. $\int \frac{dx}{\sqrt{1 + \sin x}} = \pm \sqrt{2} \log \tan \left(\frac{x}{4} + \frac{\pi}{8} \right),$
 $[\text{use} + \text{if } (8k-1)\frac{\pi}{2} < x < (8k+3)\frac{\pi}{2}, \text{ otherwise } -; k \text{ an integer}]$
420. $\int (\tan^2 ax) \, dx = \frac{1}{a} \tan ax - x$
421. $\int (\tan^3 ax) \, dx = \frac{1}{2a} \tan^2 ax + \frac{1}{a} \log \cos ax$
422. $\int (\tan^4 ax) \, dx = \frac{\tan^3 ax}{3a} - \frac{1}{a} \tan x + x$
423. $\int (\tan^n ax) \, dx = \frac{\tan^{n-1} ax}{a(n-1)} - \int (\tan^{n-2} ax) \, dx$
424. $\int (\cot^2 ax) \, dx = -\frac{1}{a} \cot ax - x$
425. $\int (\cot^3 ax) \, dx = -\frac{1}{2a} \cot^2 ax - \frac{1}{a} \log \sin ax$
426. $\int (\cot^4 ax) \, dx = -\frac{1}{3a} \cot^3 ax + \frac{1}{a} \cot ax + x$
427. $\int (\cot^n ax) \, dx = -\frac{\cot^{n-1} ax}{a(n-1)} - \int (\cot^{n-2} ax) \, dx$
428. $\int \frac{x}{\sin^2 ax} \, dx = \int x(\csc^2 ax) \, dx = -\frac{x \cot ax}{a} + \frac{1}{a^2} \log \sin ax$
429. $\int \frac{x}{\sin^n ax} \, dx = \int x(\csc^n ax) \, dx = -\frac{x \cos ax}{a(n-1)\sin^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\sin^{n-2} ax}$
 $+ \frac{(n-2)}{(n-1)} \int \frac{x}{\sin^{n-2} ax} \, dx$
430. $\int \frac{x}{\cos^2 ax} \, dx = \int x(\sec^2 ax) \, dx = \frac{1}{a} x \tan ax + \frac{1}{a^2} \log \cos ax$
431. $\int \frac{x}{\cos^n ax} \, dx = \int x(\sec^n ax) \, dx = \frac{x \sin ax}{a(n-1) \cos^{n-1} ax} - \frac{1}{a^2(n-1)(n-2) \cos^{n-2} ax}$
 $+ \frac{n-2}{n-1} \int \frac{x}{\cos^{n-2} ax} \, dx$

INTEGRALS (Continued)

432. $\int \frac{\sin ax}{\sqrt{1+b^2 \sin^2 ax}} dx = -\frac{1}{ab} \sin^{-1} \frac{b \cos ax}{\sqrt{1+b^2}}$
433. $\int \frac{\sin ax}{\sqrt{1-b^2 \sin^2 ax}} dx = -\frac{1}{ab} \log(b \cos ax + \sqrt{1-b^2 \sin^2 ax})$
434. $\int (\sin ax) \sqrt{1+b^2 \sin^2 ax} dx = -\frac{\cos ax}{2a} \sqrt{1+b^2 \sin^2 ax} - \frac{1+b^2}{2ab} \sin^{-1} \frac{b \cos ax}{\sqrt{1+b^2}}$
435. $\int (\sin ax) \sqrt{1-b^2 \sin^2 ax} dx = -\frac{\cos ax}{2a} \sqrt{1-b^2 \sin^2 ax} - \frac{1-b^2}{2ab} \log(b \cos ax + \sqrt{1-b^2 \sin^2 ax})$
436. $\int \frac{\cos ax}{\sqrt{1+b^2 \sin^2 ax}} dx = \frac{1}{ab} \log(b \sin ax + \sqrt{1+b^2 \sin^2 ax})$
437. $\int \frac{\cos ax}{\sqrt{1-b^2 \sin^2 ax}} dx = \frac{1}{ab} \sin^{-1}(b \sin ax)$
438. $\int (\cos ax) \sqrt{1+b^2 \sin^2 ax} dx = \frac{\sin ax}{2a} \sqrt{1+b^2 \sin^2 ax} + \frac{1}{2ab} \log(b \sin ax + \sqrt{1+b^2 \sin^2 ax})$
439. $\int (\cos ax) \sqrt{1-b^2 \sin^2 ax} dx = \frac{\sin ax}{2a} \sqrt{1-b^2 \sin^2 ax} + \frac{1}{2ab} \sin^{-1}(b \sin ax)$
440. $\int \frac{dx}{\sqrt{a+b \tan^2 cx}} = \frac{\pm 1}{c\sqrt{a-b}} \sin^{-1} \left(\sqrt{\frac{a-b}{a}} \sin cx \right), \quad (a > |b|)$
[use $+$ if $(2k-1)\frac{\pi}{2} < x \leq (2k+1)\frac{\pi}{2}$, otherwise $-$; k an integer]

FORMS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

441. $\int (\sin^{-1} ax) dx = x \sin^{-1} ax + \frac{\sqrt{1-a^2 x^2}}{a}$
442. $\int (\cos^{-1} ax) dx = x \cos^{-1} ax - \frac{\sqrt{1-a^2 x^2}}{a}$
443. $\int (\tan^{-1} ax) dx = x \tan^{-1} ax - \frac{1}{2a} \log(1+a^2 x^2)$
444. $\int (\cot^{-1} ax) dx = x \cot^{-1} ax + \frac{1}{2a} \log(1+a^2 x^2)$
445. $\int (\sec^{-1} ax) dx = x \sec^{-1} ax - \frac{1}{a} \log(ax + \sqrt{a^2 x^2 - 1})$
446. $\int (\csc^{-1} ax) dx = x \csc^{-1} ax + \frac{1}{a} \log(ax + \sqrt{a^2 x^2 - 1})$
447. $\int \left(\sin^{-1} \frac{x}{a} \right) dx = x \sin^{-1} \frac{x}{a} + \sqrt{a^2 - x^2}, \quad (a > 0)$
448. $\int \left(\cos^{-1} \frac{x}{a} \right) dx = x \cos^{-1} \frac{x}{a} - \sqrt{a^2 - x^2}, \quad (a > 0)$

INTEGRALS (Continued)

$$449. \int \left(\tan^{-1} \frac{x}{a} \right) dx = x \tan^{-1} \frac{x}{a} - \frac{a}{2} \log(a^2 + x^2)$$

$$450. \int \left(\cot^{-1} \frac{x}{a} \right) dx = x \cot^{-1} \frac{x}{a} + \frac{a}{2} \log(a^2 + x^2)$$

$$451. \int x [\sin^{-1}(ax)] dx = \frac{1}{4a^2} [(2a^2x^2 - 1) \sin^{-1}(ax) + ax \sqrt{1 - a^2x^2}]$$

$$452. \int x [\cos^{-1}(ax)] dx = \frac{1}{4a^2} [(2a^2x^2 - 1) \cos^{-1}(ax) - ax \sqrt{1 - a^2x^2}]$$

$$453. \int x^n [\sin^{-1}(ax)] dx = \frac{x^{n+1}}{n+1} \sin^{-1}(ax) - \frac{a}{n+1} \int \frac{x^{n+1} dx}{\sqrt{1 - a^2x^2}}, \quad (n \neq -1)$$

$$454. \int x^n [\cos^{-1}(ax)] dx = \frac{x^{n+1}}{n+1} \cos^{-1}(ax) + \frac{a}{n+1} \int \frac{x^{n+1} dx}{\sqrt{1 - a^2x^2}}, \quad (n \neq -1)$$

$$455. \int x (\tan^{-1} ax) dx = \frac{1 + a^2x^2}{2a^2} \tan^{-1} ax - \frac{x}{2a}$$

$$456. \int x^n (\tan^{-1} ax) dx = \frac{x^{n+1}}{n+1} \tan^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1}}{1 + a^2x^2} dx$$

$$457. \int x (\cot^{-1} ax) dx = \frac{1 + a^2x^2}{2a^2} \cot^{-1} ax + \frac{x}{2a}$$

$$458. \int x^n (\cot^{-1} ax) dx = \frac{x^{n+1}}{n+1} \cot^{-1} ax + \frac{a}{n+1} \int \frac{x^{n+1}}{1 + a^2x^2} dx$$

$$459. \int \frac{\sin^{-1}(ax)}{x^2} dx = a \log \left(\frac{1 - \sqrt{1 - a^2x^2}}{x} \right) - \frac{\sin^{-1}(ax)}{x}$$

$$460. \int \frac{\cos^{-1}(ax)}{x^2} dx = -\frac{1}{x} \cos^{-1}(ax) + a \log \frac{1 + \sqrt{1 - a^2x^2}}{x}$$

$$461. \int \frac{\tan^{-1}(ax)}{x^2} dx = -\frac{1}{x} \tan^{-1}(ax) - \frac{a}{2} \log \frac{1 + a^2x^2}{x^2}$$

$$462. \int \frac{\cot^{-1} ax}{x^2} dx = -\frac{1}{x} \cot^{-1} ax - \frac{a}{2} \log \frac{x^2}{a^2x^2 + 1}$$

$$463. \int (\sin^{-1} ax)^2 dx = x(\sin^{-1} ax)^2 - 2x + \frac{2\sqrt{1 - a^2x^2}}{a} \sin^{-1} ax$$

$$464. \int (\cos^{-1} ax)^2 dx = x(\cos^{-1} ax)^2 - 2x - \frac{2\sqrt{1 - a^2x^2}}{a} \cos^{-1} ax$$

$$465. \int (\sin^{-1} ax)^n dx = \begin{cases} x(\sin^{-1} ax)^n + \frac{n\sqrt{1 - a^2x^2}}{a} (\sin^{-1} ax)^{n-1} - n(n-1) \int (\sin^{-1} ax)^{n-2} dx \\ \text{or} \\ \sum_{r=0}^{[n/2]} (-1)^r \frac{n!}{(n-2r)!} x (\sin^{-1} ax)^{n-2r} \\ \quad + \sum_{r=0}^{[(n-1)/2]} (-1)^r \frac{n! \sqrt{1 - a^2x^2}}{(n-2r-1)!a} (\sin^{-1} ax)^{n-2r-1} \end{cases}$$

Note: $[s]$ means greatest integer $\leq s$. Thus $[3.5]$ means 3; $[5] = 5$, $[\frac{1}{2}] = 0$.

INTEGRALS (Continued)

$$466. \int (\cos^{-1} ax)^n dx = \begin{cases} x(\cos^{-1} ax)^n - \frac{n\sqrt{1-a^2x^2}}{a}(\cos^{-1} ax)^{n-1} - n(n-1) \int (\cos^{-1} ax)^{n-2} dx \\ \text{or} \\ \sum_{r=0}^{[n/2]} (-1)^r \frac{n!}{(n-2r)!} x(\cos^{-1} ax)^{n-2r} \\ \sum_{r=0}^{[(n-1)/2]} (-1)^r \frac{n!\sqrt{1-a^2x^2}}{(n-2r-1)!a} (\cos^{-1} ax)^{n-2r-1} \end{cases}$$

$$467. \int \frac{1}{\sqrt{1-a^2x^2}} (\sin^{-1} ax) dx = \frac{1}{2a} (\sin^{-1} ax)^2$$

$$468. \int \frac{x^n}{\sqrt{1-a^2x^2}} (\sin^{-1} ax) dx = -\frac{x^{n-1}}{na^2} \sqrt{1-a^2x^2} \sin^{-1} ax + \frac{x^n}{n^2a} \\ + \frac{n-1}{na^2} \int \frac{x^{n-2}}{\sqrt{1-a^2x^2}} \sin^{-1} ax dx$$

$$469. \int \frac{1}{\sqrt{1-a^2x^2}} (\cos^{-1} ax) dx = -\frac{1}{2a} (\cos^{-1} ax)^2$$

$$470. \int \frac{x^n}{\sqrt{1-a^2x^2}} (\cos^{-1} ax) dx = -\frac{x^{n-1}}{na^2} \sqrt{1-a^2x^2} \cos^{-1} ax - \frac{x^n}{n^2a} \\ + \frac{n-1}{na^2} \int \frac{x^{n-2}}{\sqrt{1-a^2x^2}} \cos^{-1} ax dx$$

$$471. \int \frac{\tan^{-1} ax}{a^2x^2+1} dx = \frac{1}{2a} (\tan^{-1} ax)^2$$

$$472. \int \frac{\cot^{-1} ax}{a^2x^2+1} dx = -\frac{1}{2a} (\cot^{-1} ax)^2$$

$$473. \int x \sec^{-1} ax dx = \frac{x^2}{2} \sec^{-1} ax - \frac{1}{2a^2} \sqrt{a^2x^2-1}$$

$$474. \int x^n \sec^{-1} ax dx = \frac{x^{n+1}}{n+1} \sec^{-1} ax - \frac{1}{n+1} \int \frac{x^n dx}{\sqrt{a^2x^2-1}}$$

$$475. \int \frac{\sec^{-1} ax}{x^2} dx = -\frac{\sec^{-1} ax}{x} + \frac{\sqrt{a^2x^2-1}}{x}$$

$$476. \int x \csc^{-1} ax dx = \frac{x^2}{2} \csc^{-1} ax + \frac{1}{2a^2} \sqrt{a^2x^2-1}$$

$$477. \int x^n \csc^{-1} ax dx = \frac{x^{n+1}}{n+1} \csc^{-1} ax + \frac{1}{n+1} \int \frac{x^n dx}{\sqrt{a^2x^2-1}}$$

$$478. \int \frac{\csc^{-1} ax}{x^2} dx = -\frac{\csc^{-1} ax}{x} - \frac{\sqrt{a^2x^2-1}}{x}$$

FORMS INVOLVING TRIGONOMETRIC SUBSTITUTIONS

$$479. \int f(\sin x) dx = 2 \int f\left(\frac{2z}{1+z^2}\right) \frac{dz}{1+z^2}, \quad \left(z = \tan \frac{x}{2}\right)$$

$$480. \int f(\cos x) dx = 2 \int f\left(\frac{1-z^2}{1+z^2}\right) \frac{dz}{1+z^2}, \quad \left(z = \tan \frac{x}{2}\right)$$

INTEGRALS (Continued)

$$*481. \int f(\sin x) dx = \int f(u) \frac{du}{\sqrt{1-u^2}}, \quad (u = \sin x)$$

$$*482. \int f(\cos x) dx = - \int f(u) \frac{du}{\sqrt{1-u^2}}, \quad (u = \cos x)$$

$$*483. \int f(\sin x, \cos x) dx = \int f(u, \sqrt{1-u^2}) \frac{du}{\sqrt{1-u^2}}, \quad (u = \sin x)$$

$$484. \int f(\sin x, \cos x) dx = 2 \int f\left(\frac{2z}{1+z^2}, \frac{1-z^2}{1+z^2}\right) \frac{dz}{1+z^2}, \quad \left(z = \tan \frac{x}{2}\right)$$

LOGARITHMIC FORMS

$$485. \int (\log x) dx = x \log x - x$$

$$486. \int x(\log x) dx = \frac{x^2}{2} \log x - \frac{x^2}{4}$$

$$487. \int x^2(\log x) dx = \frac{x^3}{3} \log x - \frac{x^3}{9}$$

$$488. \int x^n(\log ax) dx = \frac{x^{n+1}}{n+1} \log ax - \frac{x^{n+1}}{(n+1)^2}$$

$$489. \int (\log x)^2 dx = x(\log x)^2 - 2x \log x + 2x$$

$$490. \int (\log x)^n dx = \begin{cases} x(\log x)^n - n \int (\log x)^{n-1} dx, & (n \neq -1) \\ \text{or} \\ (-1)^n n! x \sum_{r=0}^n \frac{(-\log x)^r}{r!} \end{cases}$$

$$491. \int \frac{(\log x)^n}{x} dx = \frac{1}{n+1} (\log x)^{n+1}$$

$$492. \int \frac{dx}{\log x} = \log(\log x) + \log x + \frac{(\log x)^2}{2 \cdot 2!} + \frac{(\log x)^3}{3 \cdot 3!} + \dots$$

$$493. \int \frac{dx}{x \log x} = \log(\log x)$$

$$494. \int \frac{dx}{x(\log x)^n} = -\frac{1}{(n-1)(\log x)^{n-1}}$$

$$495. \int \frac{x^m dx}{(\log x)^n} = -\frac{x^{m+1}}{(n-1)(\log x)^{n-1}} + \frac{m+1}{n-1} \int \frac{x^m dx}{(\log x)^{n-1}}$$

$$496. \int x^m (\log x)^n dx = \begin{cases} \frac{x^{m+1} (\log x)^n}{m+1} - \frac{n}{m+1} \int x^m (\log x)^{n-1} dx \\ \text{or} \\ (-1)^n \frac{n!}{m+1} x^{m+1} \sum_{r=0}^n \frac{(-\log x)^r}{r! (m+1)^{n-r}} \end{cases}$$

$$497. \int x^p \cos(b \ln x) dx = \frac{x^{p+1}}{(p+1)^2 + b^2} \cdot [b \sin(b \ln x) + (p+1) \cos(b \ln x)] + c$$

$$498. \int x^p \sin(b \ln x) dx = \frac{x^{p+1}}{(p+1)^2 + b^2} \cdot [(p+1) \sin(b \ln x) - b \cos(b \ln x)] + c$$

$$499. \int [\log(ax+b)] dx = \frac{ax+b}{a} \log(ax+b) - x$$

* The square roots appearing in these formulas may be plus or minus, depending on the quadrant of x . Care must be used to give them the proper sign.

INTEGRALS (Continued)

$$500. \int \frac{\log(ax+b)}{x^2} dx = \frac{a}{b} \log x - \frac{ax+b}{bx} \log(ax+b)$$

$$501. \int x^m [\log(ax+b)] dx = \frac{1}{m+1} \left[x^{m+1} - \left(-\frac{b}{a}\right)^{m+1} \right] \log(ax+b) \\ - \frac{1}{m+1} \left(-\frac{b}{a}\right)^{m+1} \sum_{r=1}^{m+1} \frac{1}{r} \left(-\frac{ax}{b}\right)^r$$

$$502. \int \frac{\log(ax+b)}{x^m} dx = -\frac{1}{m-1} \frac{\log(ax+b)}{x^{m-1}} + \frac{1}{m-1} \left(-\frac{a}{b}\right)^{m-1} \log \frac{ax+b}{x} \\ + \frac{1}{m-1} \left(-\frac{a}{b}\right)^{m-1} \sum_{r=1}^{m-2} \frac{1}{r} \left(-\frac{b}{ax}\right)^r, \quad (m > 2)$$

$$503. \int \left[\log \frac{x+a}{x-a} \right] dx = (x+a) \log(x+a) - (x-a) \log(x-a)$$

$$504. \int x^m \left[\log \frac{x+a}{x-a} \right] dx = \frac{x^{m+1} - (-a)^{m+1}}{m+1} \log(x+a) - \frac{x^{m+1} - a^{m+1}}{m+1} \log(x-a) \\ + \frac{2a^{m+1}}{m+1} \sum_{r=1}^{[(m+1)/2]} \frac{1}{m-2r+2} \left(\frac{x}{a}\right)^{m-2r+2}$$

See note integral 392.

$$505. \int \frac{1}{x^2} \left[\log \frac{x+a}{x-a} \right] dx = \frac{1}{x} \log \frac{x-a}{x+a} - \frac{1}{a} \log \frac{x^2-a^2}{x^2}$$

$$506. \int (\log X) dx = \begin{cases} \left(x + \frac{b}{2c}\right) \log X - 2x + \frac{\sqrt{4ac-b^2}}{c} \tan^{-1} \frac{2cx+b}{\sqrt{4ac-b^2}}, & (b^2-4ac < 0) \\ \text{or} \\ \left(x + \frac{b}{2c}\right) \log X - 2x + \frac{\sqrt{b^2-4ac}}{c} \tanh^{-1} \frac{2cx+b}{\sqrt{b^2-4ac}}, & (b^2-4ac > 0) \\ \text{where} \\ X = a+bx+cx^2 \end{cases}$$

$$507. \int x^n (\log X) dx = \frac{x^{n+1}}{n+1} \log X - \frac{2c}{n+1} \int \frac{x^{n+2}}{X} dx - \frac{b}{n+1} \int \frac{x^{n+1}}{X} dx \\ \text{where } X = a+bx+cx^2$$

$$508. \int [\log(x^2+a^2)] dx = x \log(x^2+a^2) - 2x + 2a \tan^{-1} \frac{x}{a}$$

$$509. \int [\log(x^2-a^2)] dx = x \log(x^2-a^2) - 2x + a \log \frac{x+a}{x-a}$$

$$510. \int x [\log(x^2 \pm a^2)] dx = \frac{1}{2}(x^2 \pm a^2) \log(x^2 \pm a^2) - \frac{1}{2}x^2$$

$$511. \int [\log(x + \sqrt{x^2 \pm a^2})] dx = x \log(x + \sqrt{x^2 \pm a^2}) - \sqrt{x^2 \pm a^2}$$

$$512. \int x [\log(x + \sqrt{x^2 \pm a^2})] dx = \left(\frac{x^2}{2} \pm \frac{a^2}{4}\right) \log(x + \sqrt{x^2 \pm a^2}) - \frac{x\sqrt{x^2 \pm a^2}}{4}$$

$$513. \int x^m [\log(x + \sqrt{x^2 \pm a^2})] dx = \frac{x^{m+1}}{m+1} \log(x + \sqrt{x^2 \pm a^2}) - \frac{1}{m+1} \int \frac{x^{m+1}}{\sqrt{x^2 \pm a^2}} dx$$

INTEGRALS (Continued)

$$514. \int \frac{\log(x + \sqrt{x^2 + a^2})}{x^2} dx = -\frac{\log(x + \sqrt{x^2 + a^2})}{x} - \frac{1}{a} \log \frac{a + \sqrt{x^2 + a^2}}{x}$$

$$515. \int \frac{\log(x + \sqrt{x^2 - a^2})}{x^2} dx = -\frac{\log(x + \sqrt{x^2 - a^2})}{x} + \frac{1}{|a|} \sec^{-1} \frac{x}{a}$$

$$516. \int x^n \log(x^2 - a^2) dx = \frac{1}{n+1} \left[x^{n+1} \log(x^2 - a^2) - a^{n+1} \log(x - a) \right. \\ \left. - (-a)^{n+1} \log(x + a) - 2 \sum_{r=0}^{[n/2]} \frac{a^{2r} x^{n-2r+1}}{n-2r+1} \right]$$

See note integral 392.

EXPONENTIAL FORMS

$$517. \int e^x dx = e^x$$

$$518. \int e^{-x} dx = -e^{-x}$$

$$519. \int e^{ax} dx = \frac{e^{ax}}{a}$$

$$520. \int x e^{ax} dx = \frac{e^{ax}}{a^2} (ax - 1)$$

$$521. \int x^m e^{ax} dx = \begin{cases} \frac{x^m e^{ax}}{a} - \frac{m}{a} \int x^{m-1} e^{ax} dx \\ \text{or} \\ e^{ax} \sum_{r=0}^m (-1)^r \frac{m! x^{m-r}}{(m-r)! a^{r+1}} \end{cases}$$

$$522. \int \frac{e^{ax}}{x} dx = \log x + \frac{ax}{1!} + \frac{a^2 x^2}{2 \cdot 2!} + \frac{a^3 + x^3}{3 \cdot 3!} + \dots$$

$$523. \int \frac{e^{ax}}{x^m} dx = -\frac{1}{m-1} \frac{e^{ax}}{x^{m-1}} + \frac{a}{m-1} \int \frac{e^{ax}}{x^{m-1}} dx$$

$$524. \int e^{ax} \log x dx = \frac{e^{ax} \log x}{a} - \frac{1}{a} \int \frac{e^{ax}}{x} dx$$

$$525. \int \frac{dx}{1 + e^x} = x - \log(1 + e^x) = \log \frac{e^x}{1 + e^x}$$

$$526. \int \frac{dx}{a + b e^{px}} = \frac{x}{a} - \frac{1}{ap} \log(a + b e^{px})$$

$$527. \int \frac{dx}{a e^{mx} + b e^{-mx}} = \frac{1}{m \sqrt{ab}} \tan^{-1} \left(e^{mx} \sqrt{\frac{a}{b}} \right), \quad (a > 0, b > 0)$$

$$528. \int \frac{dx}{a e^{mx} - b e^{-mx}} = \begin{cases} \frac{1}{2m \sqrt{ab}} \log \frac{\sqrt{a} e^{mx} - \sqrt{b}}{\sqrt{a} e^{mx} + \sqrt{b}} \\ \text{or} \\ \frac{-1}{m \sqrt{ab}} \tanh^{-1} \left(\sqrt{\frac{a}{b}} e^{mx} \right), \quad (a > 0, b > 0) \end{cases}$$

$$529. \int (a^x - a^{-x}) dx = \frac{a^x + a^{-x}}{\log a}$$

$$530. \int \frac{e^{ax}}{b + c e^{ax}} dx = \frac{1}{ac} \log(b + c e^{ax})$$

$$531. \int \frac{x e^{ax}}{(1 + ax)^2} dx = \frac{e^{ax}}{a^2 (1 + ax)}$$

INTEGRALS (Continued)

$$532. \int x e^{-x^2} dx = -\frac{1}{2}e^{-x^2}$$

$$533. \int e^{ax} [\sin(bx)] dx = \frac{e^{ax} [a \sin(bx) - b \cos(bx)]}{a^2 + b^2}$$

$$534. \int e^{ax} [\sin(bx)] [\sin(cx)] dx = \frac{e^{ax} [(b-c) \sin(b-c)x + a \cos(b-c)x]}{2[a^2 + (b-c)^2]} - \frac{e^{ax} [(b+c) \sin(b+c)x + a \cos(b+c)x]}{2[a^2 + (b+c)^2]}$$

$$535. \int e^{ax} [\sin(bx)] [\cos(cx)] dx = \begin{cases} \frac{e^{ax} [a \sin(b-c)x - (b-c) \cos(b-c)x]}{2[a^2 + (b-c)^2]} \\ + \frac{e^{ax} [a \sin(b+c)x - (b+c) \cos(b+c)x]}{2[a^2 + (b+c)^2]} \\ \text{or} \\ \frac{e^{ax}}{\rho} [(a \sin bx - b \cos bx) [\cos(cx - \alpha)] \\ - c (\sin bx) \sin(cx - \alpha)] \\ \text{where} \\ \rho = \sqrt{(a^2 + b^2 - c^2)^2 + 4a^2 c^2}, \\ \rho \cos \alpha = a^2 + b^2 - c^2, \quad \rho \sin \alpha = 2ac \end{cases}$$

$$536. \int e^{ax} [\sin(bx)] [\sin(bx+c)] dx = \frac{e^{ax} \cos c}{2a} - \frac{e^{ax} [a \cos(2bx+c) + 2b \sin(2bx+c)]}{2(a^2 + 4b^2)}$$

$$537. \int e^{ax} [\sin(bx)] [\cos(bx+c)] dx = \frac{-e^{ax} \sin c}{2a} + \frac{e^{ax} [a \sin(2bx+c) - 2b \cos(2bx+c)]}{2(a^2 + 4b^2)}$$

$$538. \int e^{ax} [\cos(bx)] dx = \frac{e^{ax}}{a^2 + b^2} [a \cos(bx) + b \sin(bx)]$$

$$539. \int e^{ax} [\cos(bx)] [\cos(cx)] dx = \frac{e^{ax} [(b-c) \sin(b-c)x + a \cos(b-c)x]}{2[a^2 + (b-c)^2]} + \frac{e^{ax} [(b+c) \sin(b+c)x + a \cos(b+c)x]}{2[a^2 + (b+c)^2]}$$

$$540. \int e^{ax} [\cos(bx)] [\cos(bx+c)] dx = \frac{e^{ax} \cos c}{2a} + \frac{e^{ax} [a \cos(2bx+c) + 2b \sin(2bx+c)]}{2(a^2 + 4b^2)}$$

$$541. \int e^{ax} [\cos(bx)] [\sin(bx+c)] dx = \frac{e^{ax} \sin c}{2a} + \frac{e^{ax} [a \sin(2bx+c) - 2b \cos(2bx+c)]}{2(a^2 + 4b^2)}$$

$$542. \int e^{ax} [\sin^n bx] dx = \frac{1}{a^2 + n^2 b^2} \left[(a \sin bx - nb \cos bx) e^{ax} \sin^{n-1} bx + n(n-1)b^2 \int e^{ax} [\sin^{n-2} bx] dx \right]$$

$$543. \int e^{ax} [\cos^n bx] dx = \frac{1}{a^2 + n^2 b^2} \left[(a \cos bx + nb \sin bx) e^{ax} \cos^{n-1} bx + n(n-1)b^2 \int e^{ax} [\cos^{n-2} bx] dx \right]$$

INTEGRALS (Continued)

$$\begin{aligned}
 544. \quad \int x^m e^x \sin x \, dx &= \frac{1}{2} x^m e^x (\sin x - \cos x) \\
 &\quad - \frac{m}{2} \int x^{m-1} e^x \sin x \, dx + \frac{m}{2} \int x^{m-1} e^x \cos x \, dx \\
 545. \quad \int x^m e^{ax} [\sin bx] \, dx &= \begin{cases} x^m e^{ax} \frac{a \sin bx - b \cos bx}{a^2 + b^2} \\
 \quad - \frac{m}{a^2 + b^2} \int x^{m-1} e^{ax} (a \sin bx - b \cos bx) \, dx \\
 \text{or} \\
 e^{ax} \sum_{r=0}^m \frac{(-1)^r m! x^{m-r}}{\rho^{r+1} (m-r)!} \sin[bx - (r+1)\alpha] \\
 \text{where} \\
 \rho = \sqrt{a^2 + b^2}, \quad \rho \cos \alpha = a, \quad \rho \sin \alpha = b \end{cases} \\
 546. \quad \int x^m e^x \cos x \, dx &= \frac{1}{2} x^m e^x (\sin x + \cos x) \\
 &\quad - \frac{m}{2} \int x^{m-1} e^x \sin x \, dx - \frac{m}{2} \int x^{m-1} e^x \cos x \, dx \\
 547. \quad \int x^m e^{ax} \cos bx \, dx &= \begin{cases} x^m e^{ax} \frac{a \cos bx + b \sin bx}{a^2 + b^2} \\
 \quad - \frac{m}{a^2 + b^2} \int x^{m-1} e^{ax} (a \cos bx + b \sin bx) \, dx \\
 \text{or} \\
 e^{ax} \sum_{r=0}^m \frac{(-1)^r m! x^{m-r}}{\rho^{r+1} (m-r)!} \cos[bx - (r+1)\alpha] \\
 \rho = \sqrt{a^2 + b^2}, \quad \rho \cos \alpha = a, \quad \rho \sin \alpha = b \end{cases}
 \end{aligned}$$

INTEGRALS (Continued)

$$548. \int e^{ax}(\cos^m x)(\sin^n x) dx = \left\{ \begin{array}{l} \frac{e^{ax} \cos^{m-1} x \sin^n x [a \cos x + (m+n) \sin x]}{(m+n)^2 + a^2} \\ - \frac{na}{(m+n)^2 + a^2} \int e^{ax}(\cos^{m-1} x)(\sin^{n-1} x) dx \\ + \frac{(m-1)(m+n)}{(m+n)^2 + a^2} \int e^{ax}(\cos^{m-2} x)(\sin^n x) dx \\ \text{or} \\ \frac{e^{ax} \cos^m x [\sin^{n-1} x [a \sin x - (m+n) \cos x]}{(m+n)^2 + a^2} \\ + \frac{ma}{(m+n)^2 + a^2} \int e^{ax}(\cos^{m-1} x)(\sin^{n-1} x) dx \\ + \frac{(n-1)(m+n)}{(m+n)^2 + a^2} \int e^{ax}(\cos^m x)(\sin^{n-2} x) dx \\ \text{or} \\ \frac{e^{ax}(\cos^{m-1} x)(\sin^{n-1} x)(a \sin x \cos x + m \sin^2 x - n \cos^2 x)}{(m+n)^2 + a^2} \\ + \frac{m(m-1)}{(m+n)^2 + a^2} \int e^{ax}(\cos^{m-2} x)(\sin^n x) dx \\ + \frac{n(n-1)}{(m+n)^2 + a^2} \int e^{ax}(\cos^m x)(\sin^{n-2} x) dx \\ \text{or} \\ \frac{e^{ax}(\cos^{m-1} x)(\sin^{n-1} x)(a \cos x \sin x + m \sin^2 x - n \cos^2 x)}{(m+n)^2 + a^2} \\ + \frac{m(m-1)}{(m+n)^2 + a^2} \int e^{ax}(\cos^{m-2} x)(\sin^{n-2} x) dx \\ + \frac{(n-m)(n+m-1)}{(m+n)^2 + a^2} \int e^{ax}(\cos^m x)(\sin^{n-2} x) dx \end{array} \right.$$

$$549. \int x e^{ax}(\sin bx) dx = \frac{x e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) - \frac{e^{ax}}{(a^2 + b^2)^2} [(a^2 - b^2) \sin bx - 2ab \cos bx]$$

$$550. \int x e^{ax}(\cos bx) dx = \frac{x e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) - \frac{e^{ax}}{(a^2 + b^2)^2} [(a^2 - b^2) \cos bx + 2ab \sin bx]$$

$$551. \int \frac{e^{ax}}{\sin^n x} dx = -\frac{e^{ax}[a \sin x + (n-2) \cos x]}{(n-1)(n-2) \sin^{n-1} x} + \frac{a^2 + (n-2)^2}{(n-1)(n-2)} \int \frac{e^{ax}}{\sin^{n-2} x} dx$$

$$552. \int \frac{e^{ax}}{\cos^n x} dx = -\frac{e^{ax}[a \cos x - (n-2) \sin x]}{(n-1)(n-2) \cos^{n-1} x} + \frac{a^2 + (n-2)^2}{(n-1)(n-2)} \int \frac{e^{ax}}{\cos^{n-2} x} dx$$

$$553. \int e^{ax} \tan^n x dx = e^{ax} \frac{\tan^{n-1} x}{n-1} - \frac{a}{n-1} \int e^{ax} \tan^{n-1} x dx - \int e^{ax} \tan^{n-2} x dx$$

HYPERBOLIC FORMS

$$554. \int (\sinh x) dx = \cosh x$$

$$555. \int (\cosh x) dx = \sinh x$$

INTEGRALS (Continued)

$$556. \int (\tanh x) dx = \log \cosh x$$

$$557. \int (\coth x) dx = \log \sinh x$$

$$558. \int (\operatorname{sech} x) dx = \tan^{-1}(\sinh x)$$

$$559. \int (\operatorname{csch} x) dx = \log \tanh\left(\frac{x}{2}\right)$$

$$560. \int x(\sinh x) dx = x \cosh x - \sinh x$$

$$561. \int x^n (\sinh x) dx = x^n \cosh x - n \int x^{n-1} (\cosh x) dx$$

$$562. \int x(\cosh x) dx = x \sinh x - \cosh x$$

$$563. \int x^n (\cosh x) dx = x^n \sinh x - n \int x^{n-1} (\sinh x) dx$$

$$564. \int (\operatorname{sech} x)(\tanh x) dx = -\operatorname{sech} x$$

$$565. \int (\operatorname{csch} x)(\coth x) dx = -\operatorname{csch} x$$

$$566. \int (\sinh^2 x) dx = \frac{\sinh 2x}{4} - \frac{x}{2}$$

$$567. \int (\sinh^m x)(\cosh^n x) dx = \begin{cases} \frac{1}{m+n} (\sinh^{m+1} x)(\cosh^{n-1} x) \\ + \frac{n-1}{m+n} \int (\sinh^m x)(\cosh^{n-2} x) dx \\ \text{or} \\ \frac{1}{m+n} \sinh^{m-1} x \cosh^{n+1} x \\ - \frac{m-1}{m+n} \int (\sinh^{m-2} x)(\cosh^n x) dx, \quad (m+n \neq 0) \end{cases}$$

$$568. \int \frac{dx}{(\sinh^m x)(\cosh^n x)} \begin{cases} -\frac{1}{(m-n)(\sinh^{m-1} x)(\cosh^{n-1} x)} \\ -\frac{m+n-2}{m-1} \int \frac{dx}{(\sinh^{m-2} x)(\cosh^n x)}, \quad (m \neq 1) \\ \text{or} \\ \frac{1}{(n-1) \sinh^{m-1} x \cosh^{n-1} x} \\ + \frac{m+n-2}{n-1} \int \frac{dx}{(\sinh^m x)(\cosh^{n-2} x)}, \quad (n \neq 1) \end{cases}$$

$$569. \int (\tanh^2 x) dx = x - \tanh x$$

$$570. \int (\tanh^n x) dx = -\frac{\tanh^{n-1} x}{n-1} + \int (\tanh^{n-2} x) dx, \quad (n \neq 1)$$

$$571. \int (\operatorname{sech}^2 x) dx = \tanh x$$

$$572. \int (\cosh^2 x) dx = \frac{\sinh 2x}{4} + \frac{x}{2}$$

INTEGRALS (Continued)

573. $\int (\coth^2 x) dx = x - \coth x$
574. $\int (\coth^n x) dx = -\frac{\coth^{n-1} x}{n-1} + \int \coth^{n-2} x dx, \quad (n \neq 1)$
575. $\int (\operatorname{csch}^2 x) dx = -\operatorname{ctnh} x$
576. $\int (\sinh mx)(\sinh nx) dx = \frac{\sinh(m+n)x}{2(m+n)} - \frac{\sinh(m-n)x}{2(m-n)}, \quad (m^2 \neq n^2)$
577. $\int (\cosh mx)(\cosh nx) dx = \frac{\sinh(m+n)x}{2(m+n)} + \frac{\sinh(m-n)x}{2(m-n)}, \quad (m^2 \neq n^2)$
578. $\int (\sinh mx)(\cosh nx) dx = \frac{\cosh(m+n)x}{2(m+n)} + \frac{\cosh(m-n)x}{2(m-n)}, \quad (m^2 \neq n^2)$
579. $\int \left(\sinh^{-1} \frac{x}{a} \right) dx = x \sinh^{-1} \frac{x}{a} - \sqrt{x^2 + a^2}, \quad (a > 0)$
580. $\int x \left(\sinh^{-1} \frac{x}{a} \right) dx = \left(\frac{x^2}{2} + \frac{a^2}{4} \right) \sinh^{-1} \frac{x}{a} - \frac{x}{4} \sqrt{x^2 + a^2}, \quad (a > 0)$
581. $\int x^n (\sinh^{-1} x) dx = \left(\frac{x^{n+1}}{n+1} \right) \sinh^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{(1+x^2)^{1/2}} dx, \quad (n \neq -1)$
582. $\int \left(\cosh^{-1} \frac{x}{a} \right) dx = \begin{cases} x \cosh^{-1} \frac{x}{a} - \sqrt{x^2 - a^2}, & \left(\cosh^{-1} \frac{x}{a} > 0 \right) \\ \text{or} \\ x \cosh^{-1} \frac{x}{a} + \sqrt{x^2 - a^2}, & \left(\cosh^{-1} \frac{x}{a} < 0 \right), \end{cases} \quad (a > 0)$
583. $\int x \left(\cosh^{-1} \frac{x}{a} \right) dx = \frac{2x^2 - a^2}{4} \cosh^{-1} \frac{x}{a} - \frac{x}{4} (x^2 - a^2)^{1/2}$
584. $\int x^n (\cosh^{-1} x) dx = \frac{x^{n+1}}{n+1} \cosh^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{(x^2 - 1)^{1/2}} dx, \quad (n \neq -1)$
585. $\int \left(\tanh^{-1} \frac{x}{a} \right) dx = x \tanh^{-1} \frac{x}{a} + \frac{a}{2} \log(a^2 - x^2), \quad \left(\left| \frac{x}{a} \right| < 1 \right)$
586. $\int \left(\coth^{-1} \frac{x}{a} \right) dx = x \coth^{-1} \frac{x}{a} + \frac{a}{2} \log(x^2 - a^2), \quad \left(\left| \frac{x}{a} \right| > 1 \right)$
587. $\int x \left(\tanh^{-1} \frac{x}{a} \right) dx = \frac{x^2 - a^2}{2} \tanh^{-1} \frac{x}{a} + \frac{ax}{2}, \quad \left(\left| \frac{x}{a} \right| < 1 \right)$
588. $\int x^n (\tanh^{-1} x) dx = \frac{x^{n+1}}{n+1} \tanh^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{1-x^2} dx, \quad (n \neq -1)$
589. $\int x \left(\coth^{-1} \frac{x}{a} \right) dx = \frac{x^2 - a^2}{2} \coth^{-1} \frac{x}{a} + \frac{ax}{2}, \quad \left(\left| \frac{x}{a} \right| > 1 \right)$
590. $\int x^n (\coth^{-1} x) dx = \frac{x^{n+1}}{n+1} \coth^{-1} x + \frac{1}{n+1} \int \frac{x^{n+1}}{x^2 - 1} dx, \quad (n \neq -1)$
591. $\int (\operatorname{sech}^{-1} x) dx = x \operatorname{sech}^{-1} x + \sin^{-1} x$
592. $\int x \operatorname{sech}^{-1} x dx = \frac{x^2}{2} \operatorname{sech}^{-1} x - \frac{1}{2} \sqrt{1 - x^2}$
593. $\int x^n \operatorname{sech}^{-1} x dx = \frac{x^{n+1}}{n+1} \operatorname{sech}^{-1} x + \frac{1}{n+1} \int \frac{x^{n+1}}{(1-x^2)^{1/2}} dx, \quad (n \neq -1)$
594. $\int \operatorname{csch}^{-1} x dx = x \operatorname{csch}^{-1} x + \frac{x}{|x|} \sinh^{-1} x$

INTEGRALS (Continued)

$$595. \int x \operatorname{csch}^{-1} x \, dx = \frac{x^2}{2} \operatorname{csch}^{-1} x + \frac{1}{2} \frac{x}{|x|} \sqrt{1+x^2}$$

$$596. \int x^n \operatorname{csch}^{-1} x \, dx = \frac{x^{n+1}}{n+1} \operatorname{csch}^{-1} x + \frac{1}{n+1} \frac{x}{|x|} \int \frac{x^n}{(x^2+1)^{\frac{1}{2}}} dx, \quad (n \neq -1)$$

DEFINITE INTEGRALS

$$597. \int_0^\infty x^{n-1} e^{-x} dx = \int_0^1 \left(\log \frac{1}{x} \right)^{n-1} dx = \frac{1}{n} \prod_{m=1}^\infty \frac{\left(1 + \frac{1}{m}\right)^n}{1 + \frac{n}{m}} \\ = \Gamma(n), \quad n \neq 0, -1, -2, -3, \dots \quad (\text{Gamma Function})$$

$$598. \int_0^\infty t^n p^{-t} dt = \frac{n!}{(\log p)^{n+1}}, \quad (n = 0, 1, 2, 3, \dots \text{ and } p > 0)$$

$$599. \int_0^\infty t^{n-1} e^{-(a+1)t} dt = \frac{\Gamma(n)}{(a+1)^n}, \quad (n > 0, a > -1)$$

$$600. \int_0^1 x^m \left(\log \frac{1}{x} \right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}, \quad (m > -1, n > -1)$$

$$601. \Gamma(n) \text{ is finite if } n > 0, \Gamma(n+1) = n\Gamma(n)$$

$$602. \Gamma(n) \cdot \Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

$$603. \Gamma(n) = (n-1)! \text{ if } n = \text{integer} > 0$$

$$604. \Gamma\left(\frac{1}{2}\right) = 2 \int_0^\infty e^{-t^2} dt = \sqrt{\pi} = 1.7724538509 \dots = \left(-\frac{1}{2}\right)!$$

$$605. \Gamma\left(n + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^n} \sqrt{\pi} \quad n = 1, 2, 3, \dots$$

$$606. \Gamma\left(-n + \frac{1}{2}\right) = \frac{(-1)^n 2^n \sqrt{\pi}}{1 \cdot 3 \cdot 5 \dots (2n-1)} \quad n = 1, 2, 3, \dots$$

$$607. \int_0^1 x^{m-1} (1-x)^{n-1} dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m, n) \\ (\text{Beta function})$$

$$608. B(m, n) = B(n, m) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \text{ where } m \text{ and } n \text{ are any positive real numbers.}$$

$$609. \int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1} \frac{\Gamma(m+1) \cdot \Gamma(n+1)}{\Gamma(m+n+2)}, \quad (m > -1, n > -1, b > a)$$

$$610. \int_1^\infty \frac{dx}{x^m} = \frac{1}{m-1}, \quad [m > 1]$$

$$611. \int_0^\infty \frac{dx}{(1+x)x^p} = \pi \csc p\pi, \quad [p < 1]$$

$$612. \int_0^\infty \frac{dx}{(1-x)x^p} = -\pi \cot p\pi, \quad [p < 1]$$

$$613. \int_0^\infty \frac{x^{p-1} dx}{(1+x)} = \frac{\pi}{\sin p\pi} \\ = B(p, 1-p) = \Gamma(p)\Gamma(1-p), \quad [0 < p < 1]$$

$$614. \int_0^\infty \frac{x^{m-1} dx}{1+x^n} = \frac{\pi}{n \sin \frac{m\pi}{n}}, \quad [0 < m < n]$$

DEFINITE INTEGRALS (Continued)

615. $\int_0^\infty \frac{x^a dx}{(m+x^b)^c} = \frac{m^{(a+1-bc)/b}}{b} \left[\frac{\Gamma\left(\frac{a+1}{b}\right)\Gamma\left(c-\frac{a+1}{b}\right)}{\Gamma(c)} \right]$
 $(a > -1, b > 0, m > 0, c > \frac{a+1}{b})$
616. $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \pi$
617. $\int_0^\infty \frac{a dx}{a^2+x^2} = \frac{\pi}{2}, \quad \text{if } a > 0; 0, \text{ if } a = 0; -\frac{\pi}{2}, \text{ if } a < 0$
618. $\int_0^a (a^2-x^2)^{n/2} dx = \frac{1}{2} \int_{-a}^a (a^2-x^2)^{n/2} dx = \frac{1 \cdot 3 \cdot 5 \cdots n}{2 \cdot 4 \cdot 6 \cdots (n+1)} \cdot \frac{\pi}{2} \cdot a^{n+1} \quad (n \text{ odd})$
619. $\int_0^a x^m (a^2-x^2)^{n/2} dx = \begin{cases} \frac{1}{2} a^{m+n+1} B\left(\frac{m+1}{2}, \frac{n+2}{2}\right) \\ \text{or} \\ \frac{1}{2} a^{m+n+1} \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{m+n+3}{2}\right)} \end{cases}$
620. $\int_0^{\pi/2} (\sin^n x) dx = \begin{cases} \int_0^{\pi/2} (\cos^n x) dx \\ \text{or} \\ \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (n-1)\pi}{2 \cdot 4 \cdot 6 \cdot 8 \cdots (n)} \cdot \frac{\pi}{2}, & (n \text{ an even integer, } n \neq 0) \\ \text{or} \\ \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots (n-1)}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (n)}, & (n \text{ an odd integer, } n \neq 1) \\ \text{or} \\ \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}, & (n > -1) \end{cases}$
621. $\int_0^\infty \frac{\sin mx dx}{x} = \frac{\pi}{2}, \quad \text{if } m > 0; 0, \text{ if } m = 0; -\frac{\pi}{2}, \text{ if } m < 0$
622. $\int_0^\infty \frac{\cos x dx}{x} = \infty$
623. $\int_0^\infty \frac{\tan x dx}{x} = \frac{\pi}{2}$
624. $\int_0^\pi \sin ax \cdot \sin bx dx = \int_0^\pi \cos ax \cdot \cos bx dx = 0, \quad (a \neq b; a, b \text{ integers})$
625. $\int_0^{\pi/a} [\sin(ax)][\cos(ax)] dx = \int_0^\pi [\sin(ax)][\cos(ax)] dx = 0$
626. $\int_0^\pi [\sin(ax)][\cos(bx)] dx = \frac{2a}{a^2-b^2}, \text{ if } a-b \text{ is odd, or } 0 \text{ if } a-b \text{ is even}$
627. $\int_0^\infty \frac{\sin x \cos mx dx}{x} = 0, \quad \text{if } m < -1 \text{ or } m > 1; \frac{\pi}{4}, \text{ if } m = \pm 1; \frac{\pi}{2}, \text{ if } m^2 < 1$

DEFINITE INTEGRALS (Continued)

628. $\int_0^\infty \frac{\sin ax \sin bx}{x^2} dx = \frac{\pi a}{2}, \quad (a \leq b)$
629. $\int_0^\pi \sin^2 mx \, dx = \int_0^\pi \cos^2 mx \, dx = \frac{\pi}{2}$
630. $\int_0^\infty \frac{\sin^2(px)}{x^2} dx = \frac{\pi p}{2}$
631. $\int_0^\infty \frac{\sin x}{x^p} dx = \frac{\pi}{2\Gamma(p) \sin(p\pi/2)}, \quad 0 < p < 1$
632. $\int_0^\infty \frac{\cos x}{x^p} dx = \frac{\pi}{2\Gamma(p) \cos(p\pi/2)}, \quad 0 < p < 1$
633. $\int_0^\infty \frac{1 - \cos px}{x^2} dx = \frac{\pi p}{2}$
634. $\int_0^\infty \frac{\sin px \cos qx}{x} dx = \left\{ 0, q > p > 0; \frac{\pi}{2}, p > q > 0; \frac{\pi}{4}, p = q > 0 \right\}$
635. $\int_0^\infty \frac{\cos(mx)}{x^2 + a^2} dx = \frac{\pi}{2|a|} e^{-|ma|}$
636. $\int_0^\infty \cos(x^2) dx = \int_0^\infty \sin(x^2) dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}$
637. $\int_0^\infty \sin ax^n dx = \frac{1}{na^{1/n}} \Gamma(1/n) \sin \frac{\pi}{2n}, \quad n > 1$
638. $\int_0^\infty \cos ax^n dx = \frac{1}{na^{1/n}} \Gamma(1/n) \cos \frac{\pi}{2n}, \quad n > 1$
639. $\int_0^\infty \frac{\sin x}{\sqrt{x}} dx = \int_0^\infty \frac{\cos x}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2}}$
640. (a) $\int_0^\infty \frac{\sin^3 x}{x} dx = \frac{\pi}{4}$ (b) $\int_0^\infty \frac{\sin^3 x}{x^2} dx = \frac{3}{4} \log 3$
641. $\int_0^\infty \frac{\sin^3 x}{x^3} dx = \frac{3\pi}{8}$
642. $\int_0^\infty \frac{\sin^4 x}{x^4} dx = \frac{\pi}{3}$
643. $\int_0^{\pi/2} \frac{dx}{1 + a \cos x} = \frac{\cos^{-1} a}{\sqrt{1 - a^2}}, \quad (a < 1)$
644. $\int_0^\pi \frac{dx}{a + b \cos x} = \frac{\pi}{\sqrt{a^2 - b^2}}, \quad (a > b \geq 0)$
645. $\int_0^{2\pi} \frac{dx}{1 + a \cos x} = \frac{2\pi}{\sqrt{1 - a^2}}, \quad (a^2 < 1)$
646. $\int_0^\infty \frac{\cos ax - \cos bx}{x} dx = \log \frac{b}{a}$
647. $\int_0^{\pi/2} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{2ab}$
648. $\int_0^{\pi/2} \frac{dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} = \frac{\pi(a^2 + b^2)}{4a^3 b^3}, \quad (a, b > 0)$
649. $\int_0^{\pi/2} \sin^{n-1} x \cos^{m-1} x \, dx = \frac{1}{2} B\left(\frac{n}{2}, \frac{m}{2}\right), \quad m \text{ and } n \text{ positive integers}$

DEFINITE INTEGRALS (Continued)

$$650. \int_0^{\pi/2} (\sin^{2n+1} \theta) d\theta = \frac{2 \cdot 4 \cdot 6 \dots (2n)}{1 \cdot 3 \cdot 5 \dots (2n+1)}, \quad (n = 1, 2, 3, \dots)$$

$$651. \int_0^{\pi/2} (\sin^{2n} \theta) d\theta = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \dots (2n)} \left(\frac{\pi}{2}\right), \quad (n = 1, 2, 3, \dots)$$

$$652. \int_0^{\pi/2} \frac{x}{\sin x} dx = 2 \left\{ \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots \right\}$$

$$653. \int_0^{\pi/2} \frac{dx}{1 + \tan^m x} = \frac{\pi}{4}$$

$$654. \int_0^{\pi/2} \sqrt{\cos \theta} d\theta = \frac{(2\pi)^{\frac{3}{2}}}{[\Gamma(\frac{1}{4})]^2}$$

$$655. \int_0^{\pi/2} (\tan^h \theta) d\theta = \frac{\pi}{2 \cos\left(\frac{h\pi}{2}\right)}, \quad (0 < h < 1)$$

$$656. \int_0^\infty \frac{\tan^{-1}(ax) - \tan^{-1}(bx)}{x} dx = \frac{\pi}{2} \log \frac{a}{b}, \quad (a, b > 0)$$

657. The area enclosed by a curve defined through the equation $x^{\frac{b}{c}} + y^{\frac{b}{c}} = a^{\frac{b}{c}}$ where $a > 0$, c a positive odd integer and b a positive even integer is given by

$$\frac{\left[\Gamma\left(\frac{c}{b}\right)\right]^2}{\Gamma\left(\frac{2c}{b}\right)} \left(\frac{2ca^2}{b}\right)$$

658. $I = \iiint_R x^{h-1} y^{m-1} z^{n-1} dv$, where R denotes the region of space bounded by the co-ordinate planes and that portion of the surface $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^k = 1$, which lies in the first octant and where $h, m, n, p, q, k, a, b, c$, denote positive real numbers is given by

$$\int_0^a x^{h-1} dx \int_0^{h[1-(x/a)^p]^{1/q}} y^m dy \int_0^{c[1-(x/a)^p - (y/b)^q]^{1/k}} z^{n-1} dz = \frac{a^h b^m c^n}{pqk} \frac{\Gamma\left(\frac{h}{p}\right) \Gamma\left(\frac{m}{q}\right) \Gamma\left(\frac{n}{k}\right)}{\Gamma\left(\frac{h}{p} + \frac{m}{q} + \frac{n}{k} + 1\right)}$$

$$659. \int_0^\infty e^{-ax} dx = \frac{1}{a}, \quad (a > 0)$$

$$660. \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \log \frac{b}{a}, \quad (a, b > 0)$$

$$661. \int_0^\infty x^n e^{-ax} dx = \begin{cases} \frac{\Gamma(n+1)}{a^{n+1}}, & (n > -1, a > 0) \\ \text{or} \\ \frac{n!}{a^{n+1}}, & (a > 0, n \text{ positive integer}) \end{cases}$$

$$662. \int_0^\infty x^n \exp(-ax^p) dx = \frac{\Gamma(k)}{pa^k}, \quad \left(n > -1, p > 0, a > 0, k = \frac{n+1}{p}\right)$$

$$663. \int_0^\infty e^{-a^2 x^2} dx = \frac{1}{2a} \sqrt{\pi} = \frac{1}{2a} \Gamma\left(\frac{1}{2}\right), \quad (a > 0)$$

$$664. \int_0^\infty x e^{-x^2} dx = \frac{1}{2}$$

DEFINITE INTEGRALS (Continued)

665. $\int_0^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4}$
666. $\int_0^\infty x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}}$
667. $\int_0^\infty x^{2n+1} e^{-ax^2} dx = \frac{n!}{2a^{n+1}}, \quad (a > 0)$
668. $\int_0^1 x^m e^{-ax} dx = \frac{m!}{a^{m+1}} \left[1 - e^{-a} \sum_{r=0}^m \frac{a^r}{r!} \right]$
669. $\int_0^\infty e^{(-x^2 - a^2/x^2)} dx = \frac{e^{-2a} \sqrt{\pi}}{2}, \quad (a \geq 0)$
670. $\int_0^\infty e^{-nx} \sqrt{x} dx = \frac{1}{2n} \sqrt{\frac{\pi}{n}}$
671. $\int_0^\infty \frac{e^{-nx}}{\sqrt{x}} dx = \sqrt{\frac{\pi}{n}}$
672. $\int_0^\infty e^{-ax} (\cos mx) dx = \frac{a}{a^2 + m^2}, \quad (a > 0)$
673. $\int_0^\infty e^{-ax} (\sin mx) dx = \frac{m}{a^2 + m^2}, \quad (a > 0)$
674. $\int_0^\infty x e^{-ax} [\sin(bx)] dx = \frac{2ab}{(a^2 + b^2)^2}, \quad (a > 0)$
675. $\int_0^\infty x e^{-ax} [\cos(bx)] dx = \frac{a^2 - b^2}{(a^2 + b^2)^2}, \quad (a > 0)$
676. $\int_0^\infty x^n e^{-ax} [\sin(bx)] dx = \frac{n![(a+ib)^{n+1} - (a-ib)^{n+1}]}{2i(a^2 + b^2)^{n+1}}, \quad (i^2 = -1, a > 0)$
677. $\int_0^\infty x^n e^{-ax} [\cos(bx)] dx = \frac{n![(a-ib)^{n+1} + (a+ib)^{n+1}]}{2(a^2 + b^2)^{n+1}}, \quad (i^2 = -1, a > 0)$
678. $\int_0^\infty \frac{e^{-ax} \sin x}{x} dx = \cot^{-1} a, \quad (a > 0)$
679. $\int_0^\infty e^{-a^2 x^2} \cos bx dx = \frac{\sqrt{\pi}}{2a} \exp\left(-\frac{b^2}{4a^2}\right), \quad (ab \neq 0)$
680. $\int_0^\infty e^{-t \cos \phi} t^{b-1} [\sin(t \sin \phi)] dt = [\Gamma(b)] \sin(b\phi), \quad \left(b > 0, -\frac{\pi}{2} < \phi < \frac{\pi}{2}\right)$
681. $\int_0^\infty e^{-t \cos \phi} t^{b-1} [\cos(t \sin \phi)] dt = [\Gamma(b)] \cos(b\phi), \quad \left(b > 0, -\frac{\pi}{2} < \phi < \frac{\pi}{2}\right)$
682. $\int_0^\infty t^{b-1} \cos t dt = [\Gamma(b)] \cos\left(\frac{b\pi}{2}\right), \quad (0 < b < 1)$
683. $\int_0^\infty t^{b-1} (\sin t) dt = [\Gamma(b)] \sin\left(\frac{b\pi}{2}\right), \quad (0 < b < 1)$
684. $\int_0^1 (\log x)^n dx = (-1)^n \cdot n!$
685. $\int_0^1 \left(\log \frac{1}{x}\right)^{\frac{1}{2}} dx = \frac{\sqrt{\pi}}{2}$
686. $\int_0^1 \left(\log \frac{1}{x}\right)^{-\frac{1}{2}} dx = \sqrt{\pi}$

DEFINITE INTEGRALS (Continued)

$$687. \int_0^1 \left(\log \frac{1}{x} \right)^n dx = n!$$

$$688. \int_0^1 x \log(1-x) dx = -\frac{3}{4}$$

$$689. \int_0^1 x \log(1+x) dx = \frac{1}{4}$$

$$690. \int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}, \quad m > -1, \quad n = 0, 1, 2, \dots$$

If $n \neq 0, 1, 2, \dots$ replace $n!$ by $\Gamma(n+1)$.

$$691. \int_0^1 \frac{\log x}{1+x} dx = -\frac{\pi^2}{12}$$

$$692. \int_0^1 \frac{\log x}{1-x} dx = -\frac{\pi^2}{6}$$

$$693. \int_0^1 \frac{\log(1+x)}{x} dx = \frac{\pi^2}{12}$$

$$694. \int_0^1 \frac{\log(1-x)}{x} dx = -\frac{\pi^2}{6}$$

$$695. \int_0^1 (\log x)[\log(1+x)] dx = 2 - 2 \log 2 - \frac{\pi^2}{12}$$

$$696. \int_0^1 (\log x)[\log(1-x)] dx = 2 - \frac{\pi^2}{6}$$

$$697. \int_0^1 \frac{\log x}{1-x^2} dx = -\frac{\pi^2}{8}$$

$$698. \int_0^1 \log\left(\frac{1+x}{1-x}\right) \cdot \frac{dx}{x} = \frac{\pi^2}{4}$$

$$699. \int_0^1 \frac{\log x dx}{\sqrt{1-x^2}} = -\frac{\pi}{2} \log 2$$

$$700. \int_0^1 x^m \left[\log\left(\frac{1}{x}\right) \right]^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}, \quad \text{if } m+1 > 0, \quad n+1 > 0$$

$$701. \int_0^1 \frac{(x^p - x^q) dx}{\log x} = \log\left(\frac{p+1}{q+1}\right), \quad (p+1 > 0, \quad q+1 > 0)$$

$$702. \int_0^1 \frac{dx}{\sqrt{\log\left(\frac{1}{x}\right)}} = \sqrt{\pi}, \text{ (same as integral 686)}$$

$$703. \int_0^\infty \log\left(\frac{e^x + 1}{e^x - 1}\right) dx = \frac{\pi^2}{4}$$

$$704. \int_0^{\pi/2} (\log \sin x) dx = \int_0^{\pi/2} \log \cos x dx = -\frac{\pi}{2} \log 2$$

$$705. \int_0^{\pi/2} (\log \sec x) dx = \int_0^{\pi/2} \log \csc x dx = \frac{\pi}{2} \log 2$$

$$706. \int_0^\pi x(\log \sin x) dx = -\frac{\pi^2}{2} \log 2$$

$$707. \int_0^{\pi/2} (\sin x)(\log \sin x) dx = \log 2 - 1$$

DEFINITE INTEGRALS (Continued)

$$708. \int_0^{\pi/2} (\log \tan x) dx = 0$$

$$709. \int_0^{\pi} \log(a \pm b \cos x) dx = \pi \log \left(\frac{a + \sqrt{a^2 - b^2}}{2} \right), \quad (a \geq b)$$

$$710. \int_0^{\pi} \log(a^2 - 2ab \cos x + b^2) dx = \begin{cases} 2\pi \log a, & a \geq b > 0 \\ 2\pi \log b, & b \geq a > 0 \end{cases}$$

$$711. \int_0^{\infty} \frac{\sin ax}{\sinh bx} dx = \frac{\pi}{2b} \tanh \frac{a\pi}{2b}$$

$$712. \int_0^{\infty} \frac{\cos ax}{\cosh bx} dx = \frac{\pi}{2b} \operatorname{sech} \frac{a\pi}{2b}$$

$$713. \int_0^{\infty} \frac{dx}{\cosh ax} = \frac{\pi}{2a}$$

$$714. \int_0^{\infty} \frac{x dx}{\sinh ax} = \frac{\pi^2}{4a^2}$$

$$715. \int_0^{\infty} e^{-ax} (\cosh bx) dx = \frac{a}{a^2 - b^2}, \quad (0 \leq |b| < a)$$

$$716. \int_0^{\infty} e^{-ax} (\sinh bx) dx = \frac{b}{a^2 - b^2}, \quad (0 \leq |b| < a)$$

$$717. \int_0^{\infty} \frac{\sinh ax}{e^{bx} + 1} dx = \frac{\pi}{2b} \operatorname{csc} \frac{a\pi}{b} - \frac{1}{2a}$$

$$718. \int_0^{\infty} \frac{\sinh ax}{e^{bx} - 1} dx = \frac{1}{2a} - \frac{\pi}{2b} \cot \frac{a\pi}{b}$$

$$719. \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{\pi}{2} \left[1 + \left(\frac{1}{2} \right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 k^6 + \dots \right], \quad \text{if } k^2 < 1$$

$$720. \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 x} dx = \frac{\pi}{2} \left[1 - \left(\frac{1}{2} \right)^2 k^2 - \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \frac{k^4}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 \frac{k^6}{5} - \dots \right], \quad \text{if } k^2 < 1$$

$$721. \int_0^{\infty} e^{-x} \log x dx = -\gamma = -0.5772157 \dots$$

$$722. \int_0^{\infty} e^{-x^2} \log x dx = -\frac{\sqrt{\pi}}{4} (\gamma + 2 \log 2)$$

$$723. \int_0^{\infty} \left(\frac{1}{1 - e^{-x}} - \frac{1}{x} \right) e^{-x} dx = \gamma = 0.5772157 \dots \quad [\text{Euler's Constant}]$$

$$724. \int_0^{\infty} \frac{1}{x} \left(\frac{1}{1+x} - e^{-x} \right) dx = \gamma = 0.5772157 \dots$$

For n even:

$$725. \int \cos^n x dx = \frac{1}{2^{n-1}} \sum_{k=0}^{n/2-1} \binom{n}{k} \frac{\sin(n-2k)x}{(n-2k)} + \frac{1}{2^n} \binom{n}{n/2} x$$

DEFINITE INTEGRALS (Continued)

$$726. \int \sin^n x \, dx = \frac{1}{2^{n-1}} \sum_{k=0}^{n/2-1} \binom{n}{k} \frac{\sin \left[(n-2k) \left(\frac{\pi}{2} - x \right) \right]}{2k-n} + \frac{1}{2^n} \binom{n}{n/2} x$$

For n odd:

$$727. \int \cos^n x \, dx = \frac{1}{2^{n-1}} \sum_{k=0}^{(n-1)/2} \binom{n}{k} \frac{\sin(n-2k)x}{n-2k}$$

$$728. \int \sin^n x \, dx = \frac{1}{2^{n-1}} \sum_{k=0}^{(n-1)/2} \binom{n}{k} \frac{\sin \left[(n-2k) \left(\frac{\pi}{2} - x \right) \right]}{2k-n}$$

DIFFERENTIAL EQUATIONS

SPECIAL FORMULAS

Certain types of differential equations occur sufficiently often to justify the use of formulas for the corresponding particular solutions. The following set of tables I to XIV covers all first, second, and n th order ordinary linear differential equations with constant coefficients for which the right members are of the form $P(x)e^{rx} \sin sx$ or $P(x)e^{rx} \cos sx$, where r and s are constants and $P(x)$, is a polynomial of degree n .

When the right member of a reducible linear partial differential equation with constant coefficients is not zero, particular solutions for certain types of right members are contained in tables XV to XXI. In these tables both F and P are used to denote polynomials, and it is assumed that no denominator is zero. In any formula the roles of x and y may be reversed throughout, changing a formula in which x dominates to one in which y dominates. Tables XIX, XX, XXI are applicable whether the equations are reducible or not. The symbol $\left(\frac{m}{n}\right)$ stands for $\frac{m!}{(m-n)!n!}$ and is the $n+1$ st coefficient in the expansion of $(a+b)^m$. Also $0! = 1$ by definition.

The tables as herewith given are those contained in the text *Differential Equations* by Ginn and Company (1955) and are published with their kind permission and that of the author, Professor Frederick H. Steen.

Solution of Linear Differential Equations with Constant Coefficients

Any linear differential equation with constant coefficients may be written in the form

$$p(D)y = R(x)$$

where D is the differential operation

$$Dy = \frac{dy}{dx}$$

$p(D)$ is a polynomial in D ,

y is the dependent variable,

x is the independent variable,

$R(x)$ is an arbitrary function of x .

A power of D represents repeated differentiation, that is

$$D^n y = \frac{d^n y}{dx^n}$$

For such an equation, the general solution may be written in the form

$$y = y_c + y_p$$

where y_p is any particular solution, and y_c is called the *complementary function*. This complementary function is defined as the general solution of the *homogeneous equation*, which is the original differential equation with the right side replaced by zero, i.e.

$$p(D)y = 0$$

The complementary function y_c may be determined as follows:

1. Factor the polynomial $p(D)$ into real and complex linear factors, just as if D were a variable instead of an operator.
2. For each nonrepeated linear factor of the form $(D - a)$, where a is real, write down a term of the form

$$ce^{ax}$$

where c is an arbitrary constant.

3. For each repeated real linear factor of the form $(D - a)^n$, write down n terms of the form

$$c_1 e^{ax} + c_2 x e^{ax} + c_3 x^2 e^{ax} + \dots + c_n x^{n-1} e^{ax}$$

where the c_i 's are arbitrary constants.

4. For each non-repeated conjugate complex pair of factors of the form $(D - a + ib)(D - a - ib)$, write down 2 terms of the form

$$c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx$$

5. For each repeated conjugate complex pair of factors of the form $(D - a + ib)^n (D - a - ib)^n$, write down $2n$ terms of the form

$$c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx + c_3 x e^{ax} \cos bx + c_4 x e^{ax} \sin bx \\ + \dots + c_{2n-1} x^{n-1} e^{ax} \cos bx + c_{2n} x^{n-1} e^{ax} \sin bx$$

6. The sum of all the terms thus written down is the complementary function y_c .

To find the particular solution y_p , use the following tables, as shown in the examples. For cases not shown in the tables, there are various methods of finding y_p . The most general method is called *variation of parameters*. The following example illustrates the method:

Find y_p for $(D^2 - 4)y = e^x$.

This example can be solved most easily by use of equation 63 in the tables following. However it is given here as an example of the method of variation of parameters.

The complementary function is

$$y_c = c_1 e^{2x} + c_2 e^{-2x}$$

To find y_p , replace the constants in the complementary function with unknown functions,

$$y_p = u e^{2x} + v e^{-2x}$$

We now prepare to substitute this assumed solution into the original equation. We begin by taking all the necessary derivatives:

$$y_p = u e^{2x} + v e^{-2x} \\ y_p' = 2u e^{2x} + 2v e^{-2x} + u' e^{2x} - v' e^{-2x}$$

For each derivative of y_p except the highest, we set the sum of all the terms containing u' and v' to 0. Thus the above equation becomes

$$u' e^{2x} + v' e^{-2x} = 0 \quad \text{and} \quad y_p' = 2u e^{2x} - 2v e^{-2x}$$

Continuing to differentiate, we have

$$y_p'' = 4u e^{2x} + 4v e^{-2x} + 2u' e^{2x} - 2v' e^{-2x}$$

When we substitute into the original equation, all the terms not containing u' or v' cancel out. This is a consequence of the method by which y_p was set up.

Thus all that is necessary is to write down the terms containing u' or v' in the highest order derivative of y_p , multiply by the constant coefficient of the highest power of D in $p(D)$, and set it equal to $R(x)$. Together with the previous terms in u' and v' which were set equal to 0, this gives us as many linear equations in the first derivatives of the unknown functions as there are unknown functions. The first derivatives may then be solved for by algebra, and the unknown functions found by integration. In the present example, this becomes

$$u' e^{2x} + v' e^{-2x} = 0 \\ 2u' e^{2x} - 2v' e^{-2x} = e^x$$

We eliminate v' and u' separately, getting

$$4u' e^{2x} = e^x \\ 4v' e^{-2x} = -e^x$$

Thus

$$u' = \frac{1}{4} e^{-x} \\ v' = -\frac{1}{4} e^{3x}$$

Therefore, by integrating

$$u = -\frac{1}{4} e^{-x} \\ v = -\frac{1}{12} e^{3x}$$

A constant of integration is not needed, since we need only one particular solution. Thus

$$y_p = u e^{2x} + v e^{-2x} = -\frac{1}{4} e^{-x} e^{2x} - \frac{1}{12} e^{3x} e^{-2x} \\ = -\frac{1}{4} e^x - \frac{1}{12} e^x = -\frac{1}{3} e^x$$

and the general solution is

$$y = y_c + y_p = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3} e^x$$

The following samples illustrate the use of the tables.

Example 1. Solve $(D^2 - 4)y = \sin 3x$.

Substitution of $q = -4$, $s = 3$ in formula 24 gives

$$y_p = \frac{\sin 3x}{-9 - 4}$$

wherefore the general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{\sin 3x}{13}$$

Example 2. Obtain a particular solution of $(D^2 - 4D + 5)y = x^2 e^{3x} \sin x$.

Applying formula 40 with $a = 2$, $b = 1$, $r = 3$, $s = 1$, $P(x) = x^2$, $s + b = 2$, $s - b = 0$, $a - r = -1$, $(a - r)^2 + (s + b)^2 = 5$, $(a - r)^2 + (s - b)^2 = 1$, we have

$$\begin{aligned} y_p &= \frac{e^{3x} \sin x}{2} \left[\left(\frac{2}{5} - \frac{0}{1} \right) x^2 + \left(\frac{2(-1)2}{25} - \frac{2(-1)0}{1} \right) 2x + \left(\frac{3 \cdot 1 \cdot 2 - 2^3}{125} - \frac{3 \cdot 1 \cdot 0 - 0}{1} \right) 2 \right] \\ &\quad - \frac{e^{3x} \cos x}{2} \left[\left(\frac{-1}{5} - \frac{-1}{1} \right) x^2 + \left(\frac{1-4}{25} - \frac{1-0}{1} \right) 2x + \left(\frac{-1-3(-1)4}{125} - \frac{-1-3(-1)0}{1} \right) 2 \right] \\ &= \left(\frac{1}{5} x^2 - \frac{4}{25} x - \frac{2}{125} \right) e^{3x} \sin x + \left(-\frac{2}{5} x^2 + \frac{28}{25} x - \frac{136}{125} \right) e^{3x} \cos x \end{aligned}$$

The special formulas effect a very considerable saving of time in problems of this type.

Example 3. Obtain a particular solution of $(D^2 - 4D + 5)y = x^2 e^{2x} \cos x$. (Compare with Example 2.)

Formula 40 is not applicable here since for this equation $r = a$, $s = b$, wherefore the denominator $(a - r)^2 + (s - b)^2 = 0$. We turn instead to formula 44. Substituting $a = 2$, $b = 1$, $P(x) = x^2$ and replacing $\sin$ by $\cos$, $\cos$ by $-\sin$, we obtain

$$\begin{aligned} y_p &= \frac{e^{2x} \cos x}{4} \left(x^2 - \frac{2}{4} \right) + \frac{e^{2x} \sin x}{2} \int \left(x^2 - \frac{1}{2} \right) dx \\ &= \left(\frac{x^2}{4} - \frac{1}{8} \right) e^{2x} \cos x + \left(\frac{x^3}{6} - \frac{x}{4} \right) e^{2x} \sin x \end{aligned}$$

which is the required solution.

Example 4. Find z_p for $(D_x - 3D_y)z = \ln(y + 3x)$.

Referring to Table XV we note that formula 69 (not 68) is applicable. This gives

$$z_p = x \ln(y + 3x)$$

It is easily seen that $-y/3 \ln(y + 3x)$ would serve equally well.

Example 5. Solve $(D_x + 2D_y - 4)z = y \cos(y - 2x)$.

Since R in formula 76 contains a polynomial in x , not y , we rewrite the given equation in the form

$$(D_y + \frac{1}{2} D_x - 2)z = \frac{1}{2} y \cos(y - 2x)$$

Then

$$z_c = e^{2y} F\left(x - \frac{2}{1} y\right) = e^{2x} f(2x - y)$$

and by the formula

$$\begin{aligned} z_p &= -\frac{1}{2} \cos(y - 2x) \cdot \left(\frac{y}{2} + \frac{\frac{1}{2}}{2} \right) \\ &= -\frac{1}{8} (2y + 1) \cos(y - 2x) \end{aligned}$$

Example 6. Find z_p for $(D_x + 4D_y)^3 z = (2x - y)^2$.

Using formula 79, we obtain

$$z_p = \frac{\iiint u^2 du^3}{[2 + 4(-1)]^3} = \frac{u^5}{5 \cdot 4 \cdot 3 \cdot (-8)} = -\frac{(2x - y)^5}{480}$$

Example 7. Find z_p for $(D_x^3 + 5D_x^2 D_y - 7D_x + 4)z = e^{2x+3y}$.

By formula 87

$$z_p = \frac{e^{2x+3y}}{2^3 + 5 \cdot 2^2 \cdot 3 - 7 \cdot 2 + 4} = \frac{e^{2x+3y}}{58}$$

Example 8. Find z_p for

$$(D_x^4 + 6D_x^3 D_y + D_x D_y + D_y^2 + 9)z = \sin(3x + 4y)$$

Since every term in the left member is of even degree in the two operators D_x and D_y , formula 90 is applicable. It gives

$$\begin{aligned} z_p &= \frac{\sin(3x + 4y)}{(-9)^2 + 6(-9)(-12) + (-12) + (-16) + 9} \\ &= \frac{\sin(3x + 4y)}{710} \end{aligned}$$

TABLE I: $(D - a)y = R$

R	y_p
1. e^{rx}	$\frac{e^{rx}}{r - a}$
2. $\sin sx^*$	$-\frac{a \sin sx + s \cos sx}{a^2 + s^2} = \frac{1}{\sqrt{a^2 + s^2}} \sin\left(sx + \tan^{-1} \frac{s}{a}\right)$
3. $P(x)$	$-\frac{1}{a} \left[P(x) + \frac{P'(x)}{a} + \frac{P''(x)}{a^2} + \dots + \frac{P^{(n)}(x)}{a^n} \right]$
4. $e^{rx} \sin sx^*$	Replace a by $a - r$ in formula 2 and multiply by e^{rx} .
5. $P(x) e^{rx}$	Replace a by $a - r$ in formula 3 and multiply by e^{rx} .
6. $P(x) \sin sx^*$	$-\sin sx \left[\frac{a}{a^2 + s^2} P(x) + \frac{a^2 - s^2}{(a^2 + s^2)^2} P'(x) + \frac{a^3 - 3as^2}{(a^2 + s^2)^3} P''(x) + \dots + \frac{a^k - \binom{k}{2} a^{k-2} s^2 + \binom{k}{4} a^{k-4} s^4 - \dots}{(a^2 + s^2)^k} P^{(k-1)}(x) + \dots \right]$ $-\cos sx \left[\frac{s}{a^2 + s^2} P(x) + \frac{2as}{(a^2 + s^2)^2} P'(x) + \frac{3a^2 s - s^3}{(a^2 + s^2)^3} P''(x) + \dots + \frac{\binom{k}{1} a^{k-1} s - \binom{k}{3} a^{k-3} s^3 + \dots}{(a^2 + s^2)^k} P^{(k-1)}(x) + \dots \right]$
7. $P(x) e^{rx} \sin sx^*$	Replace a by $a - r$ in formula 6 and multiply by e^{rx} .
8. e^{ax}	$x e^{ax}$
9. $e^{ax} \sin sx^*$	$-\frac{e^{ax} \cos sx}{s}$
10. $P(x) e^{ax}$	$e^{ax} \int P(x) dx$
11. $P(x) e^{ax} \sin sx$	$\frac{e^{ax} \sin sx}{s} \left[\frac{P'(x)}{s^3} - \frac{P''(x)}{s^3} + \frac{P'''(x)}{s^5} - \dots \right] - \frac{e^{ax} \cos sx}{s} \left[P(x) - \frac{P'(x)}{s^2} + \frac{P''(x)}{s^4} - \dots \right]$

*For $\cos sx$ in R replace "sin" by "cos" and "cos" by "- sin" in y_p .

$$D^n = \frac{d^n}{dx^n} \quad \binom{m}{n} = \frac{m!}{(m-n)!n!} \quad 0! = 1$$

TABLE II: $(D - a)^2 y = R$

R	y_p
12. e^{rx}	$\frac{e^{rx}}{(r - a)^2}$
13. $\sin sx^*$	$\frac{1}{(a^2 + s^2)} [(a^2 - s^2) \sin sx + 2as \cos sx] = \frac{1}{a^2 + s^2} \sin\left(sx + \tan^{-1} \frac{2as}{a^2 - s^2}\right)$
14. $P(x)$	$\frac{1}{a^2} \left[P(x) + \frac{2P'(x)}{a} + \frac{3P''(x)}{a^2} + \dots + \frac{(n+1)P^{(n)}(x)}{a^n} \right]$
15. $e^{rx} \sin sx^*$	Replace a by $a - r$ in formula 13 and multiply by e^{rx} .
16. $P(x) e^{rx}$	Replace a by $a - r$ in formula 14 and multiply by e^{rx} .
17. $P(x) \sin sx^*$	$\sin sx \left[\frac{a^2 - s^2}{(a^2 + s^2)^2} P(x) + 2 \frac{a^3 - 3as^2}{(a^2 + s^2)^3} P'(x) + 3 \frac{a^4 - 6a^2 s^2 + s^4}{(a^2 + s^2)^4} P''(x) + \dots \right]$ $+ (k-1) \frac{a^k - \binom{k}{2} a^{k-2} s^2 + \binom{k}{4} a^{k-4} s^4 - \dots}{(a^2 + s^2)^k} P^{(k-2)}(x) + \dots$ $+\cos sx \left[\frac{2as}{(a^2 + s^2)^2} P(x) + 2 \frac{3a^2 s - s^3}{(a^2 + s^2)^3} P'(x) + 3 \frac{4a^3 s - 4as^3}{(a^2 + s^2)^4} P''(x) + \dots \right]$ $+ (k-1) \frac{\binom{k}{1} a^{k-1} s - \binom{k}{3} a^{k-3} s^3 + \dots}{(a^2 + s^2)^k} P^{(k-2)}(x) + \dots$
18. $P(x) e^{rx} \sin sx^*$	Replace a by $a - r$ in formula 17 and multiply by e^{rx} .
19. e^{ax}	$\frac{1}{2} x^2 e^{ax}$
20. $e^{ax} \sin sx^*$	$-\frac{e^{ax} \sin sx}{s^2}$
21. $P(x) e^{ax}$	$e^{ax} \int \int P(x) dx dx$
22. $P(x) e^{ax} \sin sx^*$	$-\frac{e^{ax} \sin sx}{s^2} \left[P(x) - \frac{3P'(x)}{s^2} + \frac{5P''(x)}{s^4} - \frac{7P'''(x)}{s^6} + \dots \right] - \frac{e^{ax} \cos sx}{s^2} \left[\frac{2P'(x)}{s} + \frac{4P''(x)}{s^3} - \frac{6P'''(x)}{s^5} - \dots \right]$

*For $\cos sx$ in R replace "sin" by "cos" by "- sin" in y_p .

DIFFERENTIAL EQUATIONS (Continued)

TABLE III: $(D^2 + q)y = R$

R	y_p
23. e^{rx}	$\frac{e^{rx}}{r^2 + q}$
24. $\sin sx^*$	$\frac{\sin sx}{-s^2 + q}$
25. $P(x)$	$\frac{1}{q} \left[P(x) - \frac{P'(x)}{q} + \frac{P^{iv}(x)}{q^2} - \dots + (-1)^k \frac{P^{(2k)}(x)}{q^k} \dots \right]$
26. $e^{rx} \sin sx$	$\frac{(r^2 - s^2 + q)e^{rx} \sin sx - 2rs e^{rx} \cos sx}{(r^2 - s^2 + q)^2 + (2rs)^2} = \frac{e^{rx}}{\sqrt{(r^2 - s^2 + q)^2 + (2rs)^2}} \sin \left[sx - \tan^{-1} \frac{2rs}{r^2 - s^2 + q} \right]$
27. $P(x) e^{rx}$	$\frac{e^{rx}}{r^2 + q} \left[P(x) - \frac{2r}{r^2 + q} P'(x) + \frac{3r^2 - q}{(r^2 + q)^2} P''(x) - \frac{4r^3 - 4qr}{(r^2 + q)^3} P'''(x) + \dots \right. \\ \left. + \dots + (-1)^{k-1} \frac{\binom{k}{1} r^{k-1} - \binom{k}{3} r^{k-3} q + \binom{k}{5} r^{k-5} q^2 - \dots}{(r^2 + q)^{k-1}} P^{(k-1)}(x) + \dots \right]$
28. $P(x) \sin sx^*$	$\frac{\sin sx}{(-s^2 + q)} \left[P(x) - \frac{3s^2 + q}{(-s^2 + q)^2} P''(x) + \frac{5s^4 + 10s^2 q + q^2}{(-s^2 + q)^4} P^{iv}(x) + \dots \right. \\ \left. + (-1)^k \frac{\binom{2k+1}{1} s^{2k} + \binom{2k+1}{3} s^{2k-2} q + \binom{2k+1}{5} s^{2k-4} q^2 + \dots}{(-s^2 + q)^{2k}} P^{(2k)}(x) + \dots \right] \\ - \frac{s \cos sx}{(-s^2 + q)} \left[\frac{2P'(x)}{(-s^2 + q)} - \frac{4s^2 + 4q}{(-s^2 + q)^3} P'''(x) + \dots + (-1)^{k+1} \frac{\binom{2k}{1} s^{2k-2} + \binom{2k}{3} s^{2k-4} q + \dots}{(-s^2 + q)^{2k-1}} P^{(2k-1)}(x) + \dots \right]$

TABLE IV: $(D^2 + b^2)y = R$

29. $\sin bx^*$	$-\frac{x \cos bx}{2b}$
30. $P(x) \sin bx^*$	$\frac{\sin bx}{(2b)^2} \left[P(x) - \frac{P''(x)}{(2b)^2} + \frac{P^{iv}(x)}{(2b)^4} - \dots \right] - \frac{\cos bx}{2b} \int \left[P(x) - \frac{P''(x)}{(2b)^2} + \dots \right] dx$

* For $\cos sx$ in R replace "sin" by "cos" and "cos" by "-sin" in y_p .

TABLE V: $(D^2 + pD + q)y = R$

R	y_p
31. e^{rx}	$\frac{e^{rx}}{r^2 + pr + q}$
32. $\sin sx^*$	$\frac{(q - s^2) \sin sx - ps \cos sx}{(q - s^2)^2 + (ps)^2} = \frac{1}{\sqrt{(q - s^2)^2 + (ps)^2}} \sin \left(sx - \tan^{-1} \frac{ps}{q - s^2} \right)$
33. $P(x)$	$\frac{1}{q} \left[P(x) - \frac{p}{q} P'(x) + \frac{p^2 - q}{q^2} P''(x) - \frac{p^3 - 2pq}{q^3} P'''(x) + \dots + (-1)^n \frac{p^n - \binom{n-1}{1} p^{n-2} q + \binom{n-2}{2} p^{n-4} q^2 - \dots}{q^n} P^{(n)}(x) \right]$
34. $e^{rx} \sin sx^*$	Replace p by $p + 2r$, q by $q + pr + r^2$ in formula 32 and multiply by e^{rx} .
35. $P(x) e^{rx}$	Replace p by $p + 2r$, q by $q + pr + r^2$ in formula 33 and multiply by e^{rx} .

TABLE VI: $(D - b)(D - a)y = R$

36. $P(x) \sin sx^*$	$\frac{\sin sx}{b - a} \left[\left(\frac{a}{a^2 + s^2} - \frac{b}{b^2 + s^2} \right) P(x) + \left(\frac{a^2 - s^2}{(a^2 + s^2)^2} - \frac{b^2 - s^2}{(b^2 + s^2)^2} \right) P'(x) + \left(\frac{a^3 - 3as^2}{(a^2 + s^2)^3} - \frac{b^3 - 3bs^2}{(b^2 + s^2)^3} \right) P''(x) + \dots \right. \\ \left. + \frac{\cos sx}{b - a} \left[\left(\frac{s}{a^2 + s^2} - \frac{s}{b^2 + s^2} \right) P(x) + \left(\frac{2as}{(a^2 + s^2)^2} - \frac{2bs}{(b^2 + s^2)^2} \right) P'(x) \right. \right. \\ \left. \left. + \left(\frac{3a^2 s - s^2}{(a^2 + s^2)^3} - \frac{3b^2 s - s^3}{(b^2 + s^2)^3} \right) P''(x) + \dots \right] \right]^\dagger$
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37. $P(x) e^{rx} \sin sx^*$ Replace a by $a - r$, b by $b - r$ in formula 36 and multiply by e^{rx} .

38. $P(x) e^{ax}$ $\frac{e^{ax}}{a - b} \left[\int P(x) dx + \frac{P(x)}{(b - a)} + \frac{P'(x)}{(b - a)^2} + \frac{P''(x)}{(b - a)^3} + \dots + \frac{P^{(n)}(x)}{(b - a)^{n+1}} \right]$

*For $\cos sx$ in R replace "sin" by "cos" and "cos" by "-sin" in y_p .

†For additional terms, compare with formula 6.

DIFFERENTIAL EQUATIONS (Continued)

TABLE VII: $(D^2 - 2aD + a^2 + b^2)y = R$

R	y_p
39. $P(x) \sin sx^*$	$\frac{\sin sx}{2b} \left[\left(\frac{s+b}{a^2 + (s+b)^2} - \frac{s-b}{a^2 + (s-b)^2} \right) P(x) + \left(\frac{2a(s+b)}{[a^2 + (s+b)^2]^2} - \frac{2a(s-b)}{[a^2 + (s-b)^2]^2} \right) P'(x) \right.$ $\left. + \left(\frac{3a^2(s+b) - (s+b)^3}{[a^2 + (s+b)^2]^3} - \frac{3a^2(s-b) - (s-b)^3}{[a^2 + (s-b)^2]^3} \right) P''(x) + \dots \right]$ $- \frac{\cos sx}{2b} \left[\left(\frac{a}{a^2 + (s+b)^2} - \frac{a}{a^2 + (s-b)^2} \right) P(x) + \left(\frac{a^2 - (s+b)^2}{[a^2 + (s+b)^2]^2} - \frac{a^2 - (s-b)^2}{[a^2 + (s-b)^2]^2} \right) P'(x) \right.$ $\left. + \left(\frac{a^2 - 3a(s+b)^2}{[a^2 + (s+b)^2]^3} - \frac{a^2 - 3a(s-b)^2}{[a^2 + (s-b)^2]^3} \right) P''(x) + \dots \right]^\dagger$

40. $P(x)e^{rx} \sin sx^*$

Replace a by $a-r$ in formula 39 and multiply by e^{rx} .

41. $P(x)e^{ax}$

$$\frac{e^{ax}}{b^2} \left[P(x) - \frac{P'(x)}{b^2} + \frac{P''(x)}{b^4} - \dots \right]$$

42. $e^{ax} \sin sx^*$

$$\frac{e^{ax} \sin sx}{-s^2 + b^2}$$

43. $e^{ax} \sin bx^*$

$$-\frac{xe^{ax} \cos bx}{2b}$$

44. $P(x)e^{ax} \sin bx^*$

$$\frac{e^{ax} \sin bx}{(2b)^2} \left[P(x) - \frac{P'(x)}{(2b)^2} + \frac{P''(x)}{(2b)^4} - \dots \right] - \frac{e^{ax} \cos bx}{2b} \int \left[P(x) - \frac{P'(x)}{(2b)^2} + \frac{P''(x)}{(2b)^4} - \dots \right] dx$$

*For $\cos sx$ in R replace “sin” by “cos” and “cos” by “-sin” in y_p .

†For additional terms, compare with formula 6.

TABLE VIII: $f(D)y = [D^n + a_{n-1}D^{n-1} + \dots + a_1D + a_0]y = R$

R	y_p
45. e^{rx}	$\frac{e^{rx}}{f(r)}$
46. $\sin sx^*$	$\frac{[a_0 - a_2s^2 + a_4s^4 - \dots] \sin sx - [a_1s - a_3s^3 + a_5s^5 - \dots] \cos sx}{[a_0 - a_2s^2 + a_4s^4 - \dots]^2 + [a_1s - a_3s^3 + a_5s^5 - \dots]^2}$

TABLE IX: $f(D^2)y = R$

47. $\sin sx^*$	$\frac{\sin sx}{f(-s^2)} = \frac{\sin sx}{a_0 - a_2s^2 + \dots \pm s^{2n}}$
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TABLE X: $(D-a)^ny = R$

48. e^{rx}	$\frac{e^{rx}}{(r-a)^n}$
49. $\sin sx^*$	$\frac{(-1)^n}{(a^2 + s^2)^2} \{ [a^n - \binom{n}{2}a^{n-2}s^2 + \binom{n}{4}a^{n-4}s^4 - \dots] \sin sx + [\binom{n}{1}a^{n-1}s - \binom{n}{3}a^{n-3}s^3 + \dots] \cos sx \}$
50. $P(x)$	$\frac{(-1)^n}{a^n} \left[P(x) + \binom{n}{1} \frac{P'(x)}{a} + \binom{n+1}{2} \frac{P''(x)}{a^2} + \binom{n+2}{3} \frac{P'''(x)}{a^3} + \dots \right]$
51. $e^{rx} \sin sx^*$	Replace a by $a-r$ in formula 49 and multiply by e^{rx} .
52. $e^{rx} P(x)$	Replace a by $a-r$ in formula 50 and multiply by e^{rx} .
53. $P(x) \sin sx^*$	$(-1)^n \sin sx [A_n P(x) + \binom{n}{1} A_{n+1} P'(x) + \binom{n+1}{2} A_{n+2} P''(x) + \binom{n+2}{3} A_{n+3} P'''(x) + \dots]$ $+ (-1)^n \cos sx [B_n P(x) + \binom{n}{1} B_{n+1} P'(x) + \binom{n+1}{2} B_{n+2} P''(x) + \binom{n+2}{3} B_{n+3} P'''(x) + \dots]$ $A_1 = \frac{a}{a^2 + s^2}, A_2 = \frac{a^2 - s^2}{(a^2 + s^2)^2}, \dots, A_k = \frac{a^k - \binom{k}{2}a^{k-2}s^2 + \binom{k}{4}a^{k-4}s^4 - \dots}{(a^2 + s^2)^k}$ $B_1 = \frac{a}{a^2 + s^2}, B_2 = \frac{2as}{(a^2 + s^2)^2}, \dots, B_k = \frac{\binom{k}{1}a^{k-1}s - \binom{k}{3}a^{k-3}s^3 + \dots}{(a^2 + s^2)^k}$
54. $e^{rx} \sin sx^*$	Replace a by $a-r$ in formula 53 and multiply by e^{rx} .

DIFFERENTIAL EQUATIONS (Continued)

$$\begin{aligned}
 55. \quad & e^{ax} P(x) \quad e^{ax} \int \dots \int P(x) dx^n \\
 56. \quad & P(x) e^{ax} \sin sx^* \quad \frac{(-1)^{(n-1)/2} e^{ax} \sin sx}{s^n} \left[\binom{n}{n-1} \frac{P'(x)}{s} - \binom{n+2}{n-1} \frac{P''(x)}{s^3} + \binom{n+4}{n-1} \frac{P'''(x)}{s^5} - \dots \right] \\
 & \quad + \frac{(-1)^{(n+1)/2} e^{ax} \cos sx}{s^n} \left[\binom{n-1}{n-1} P(x) - \binom{n+1}{n-1} \frac{P'(x)}{s^2} + \binom{n+3}{n-1} \frac{P''(x)}{s^4} - \dots \right] \quad (n \text{ odd}) \\
 & \quad \frac{(-1)^{n/2} e^{ax} \sin sx}{s^n} \left[\binom{n-1}{n-1} P(x) - \binom{n+1}{n-1} \frac{P'(x)}{s^2} + \binom{n+3}{n-1} \frac{P''(x)}{s^4} - \dots \right] \\
 & \quad + \frac{(-1)^{n/2} e^{ax} \cos sx}{s^n} \left[\binom{n}{n-1} \frac{P'(x)}{s} - \binom{n+2}{n-1} \frac{P''(x)}{s^3} + \binom{n+4}{n-1} \frac{P'''(x)}{s^5} - \dots \right] \quad (n \text{ even})
 \end{aligned}$$

*For $\cos sx$ in R replace “sin” by “cos” and “cos” by “-sin” in y_p .

TABLE XI: $(D - a)^n f(D)y = R$

$$\begin{aligned}
 57. \quad & e^{ax} \quad \frac{x^n}{n!} \cdot \frac{e^{ax}}{f(a)} \\
 & \text{*For } \cos sx \text{ in } R \text{ replace “sin” by “cos” and “cos” by “-sin” in } y_p.
 \end{aligned}$$

TABLE XII: $(D^2 + q)^n y = R$

$$\begin{aligned}
 R & \quad y_p \\
 58. \quad & e^{rx} \quad e^{rx} / (r^2 + q)^n \\
 59. \quad & \sin sx^* \quad \sin sx / (q - s^2)^n \\
 60. \quad & P(x) \quad \frac{1}{q^n} \left[P(x) - \binom{n}{1} \frac{P'(x)}{q^2} + \binom{n+1}{2} \frac{P''(x)}{q^2} - \binom{n+2}{3} \frac{P'''(x)}{q^3} + \dots \right] \\
 61. \quad & e^{rx} \sin sx^* \quad \frac{e^{rx}}{(A^2 + B^2)^n} \{ [A^n - \binom{n}{2} A^{n-2} B^2 + \binom{n}{4} A^{n-4} B^4 - \dots] \sin sx - [\binom{n}{1} A^{n-1} B - \binom{n}{3} A^{n-3} B^3 + \dots] \cos sx \} \\
 & \quad A = r^2 - s^2 + q, \quad B = 2rs
 \end{aligned}$$

TABLE XIII: $(D^2 + b^2)^n y = R$

$$62. \quad \sin bx^* \quad (-1)^{(n+1)/2} \frac{x^n \cos bx}{n!(2b)^n} \quad (n \text{ odd}), \quad (-1)^{n/2} \frac{x^n \sin bx}{n!(2b)^n} \quad (n \text{ even})$$

TABLE XIV: $(D^n - q)y = R$

$$\begin{aligned}
 63. \quad & e^{rx} \quad e^{rx} / (r^n - q) \\
 64. \quad & P(x) \quad -\frac{1}{q} \left[P(x) \frac{P^{(n)}(x)}{q} + \frac{P^{(2n)}(x)}{q^2} + \dots \right] \\
 65. \quad & \sin sx^* \quad -\frac{q \sin sx + (-1)^{(n-1)/2} s^n \cos sx}{q^2 + s^{2n}} \quad (n \text{ odd}), \quad \frac{\sin sx}{(-s^2)^{n/2} - q} \quad (n \text{ even}) \\
 66. \quad & e^{rx} \sin sx^* \quad \frac{A e^{rx} \sin sx - B e^{rx} \cos sx}{A^2 + B^2} = \frac{e^{rx}}{\sqrt{A^2 + B^2}} \sin \left(sx - \tan^{-1} \frac{B}{A} \right) \\
 & \quad A = [r^n - \binom{n}{2} r^{n-2} s^2 + \binom{n}{4} r^{n-4} s^4 - \dots] - q, \quad B = [\binom{n}{1} r^{n-1} s - \binom{n}{3} r^{n-3} s^3 + \dots] \\
 & \quad \text{*For } \cos sx \text{ in } R \text{ replace “sin” by “cos” and “cos” by “- sin” in } y_p.
 \end{aligned}$$

TABLE XV: $(D_x + mD_y)z = R$

$$\begin{aligned}
 R & \quad z_p \\
 67. \quad & e^{ax+by} \quad \frac{e^{ax+by}}{a+mb} \\
 68. \quad & f(ax+by) \quad \frac{\int f(u) du}{a+mb}, \quad u = ax+by \\
 69. \quad & f(y-mx) \quad xf(y-mx) \\
 70. \quad & \phi(x,y)f(y-mx) \quad f(y-mx) \int \phi(x, a+mx) dx \quad (a = y-mx \text{ after integration})
 \end{aligned}$$

DIFFERENTIAL EQUATIONS (Continued)

TABLE XVI: $(D_x + mD_y - k)z = R$

71. e^{ax+by}	$\frac{e^{ax+by}}{a+mb-k}$
72. $\sin(ax+by)^*$	$-\frac{(a+bm)\cos(ax+by)+k\sin(ax+by)}{(a+bm)^2+k^2}$
73. $e^{\alpha x+\beta y}\sin(ax+by)^*$	Replace k in 72 by $k-\alpha-m\beta$ and multiply by $e^{\alpha x+\beta y}$
74. $e^{kx}f(ax+by)$	$\frac{e^{kx}\int f(u)du}{\frac{q+mb}{f(y-mx)}}, \quad u=ax+by$
75. $f(y-mx)$	$-\frac{k}{f(y-mx)}$
76. $p(x)f(y-mx)$	$-\frac{1}{k}f(y-mx)\left[p(x)+\frac{P'(x)}{k}+\frac{P''(x)}{k^2}+\cdots+\frac{P^{(n)}(x)}{k^n}\right]$
77. $e^{kx}f(y-mx)$	$xe^{kx}f(y-mx)$ *For $\cos(ax+by)$ replace “sin” by “cos” and “cos” by “-sin” in z_p .
	$D_x = \frac{\partial}{\partial x}; \quad D_y = \frac{\partial}{\partial y}; \quad D_x^* D_y^* = \frac{d^{k+r}}{\partial x^* \partial y^*}$

TABLE XVII: $(D_z + mD_y)^n z = R$

R	z_p
78. e^{ax+by}	$\frac{e^{ax+by}}{(a+mb)^n}$
79. $f(ax+by)$	$\frac{\int \int \cdots \int f(u) du^n}{(a+mb)^n}, \quad u=ax+by$
80. $f(y-mx)$	$\frac{x^n}{n!}f(y-mx)$
81. $\phi(x,y)f(y+mx)$	$f(y-mx) \int \int \cdots \int \phi(x, a+mx) dx^n \quad (a=y-mx \text{ after integration})$

TABLE XVIII: $(D_x + mD_y - k)^n z = R$

82. e^{ax+by}	$\frac{e^{ax+by}}{(a+mb-k)^n}$
83. $f(y-mx)$	$\frac{(-1)^n f(y-mx)}{k^n}$
84. $P(x)f(y-mx)$	$\frac{(-1)^n}{k^n}f(y-mx)\left[P(x)+\binom{n}{1}\frac{P'(x)}{k}+\binom{n+1}{2}\frac{P''(x)}{k^2}+\binom{n+2}{3}\frac{P'''(x)}{k^3}+\cdots\right]$
85. $e^{kz}f(ax+by)$	$\frac{e^{kx}\int \int \cdots \int f(u) du^n}{(a+mb)^n}, \quad u=ax+by$
86. $e^{kx}f(y-mx)$	$\frac{x^n}{n!}e^{kx}f(y-mx)$

TABLE XIX: $\left[D_x^n + a_1 D_x^{n-1} D_y + a_2 D_x^{n-2} D_y^2 + \cdots + a^n D_y^n\right]z = R$

87. e^{ax+by}	$\frac{e^{ax+by}}{a+a_1 a^{n-1} b+a_2 a^{n-2} b^2+\cdots+a_n b^n}$
88. $f(ax+by)$	$\frac{\int \int \cdots \int f(u) du^n}{a^n+a_1 a^{n-1} b+a_2 a^{n-2} b^2+\cdots+a_n b^n}, \quad (u=ax+by)$

TABLE XX: $F(D_x, D_y)z = R$

89. e^{ax+by}	$\frac{e^{ax+by}}{F(a, b)}$
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TABLE XXI: $F\left(D_x^2, D_x D_y, D_y^2\right)z = R$

90. $\sin(ax+by)^*$	$\frac{\sin(ax+by)}{F(-a^2, -ab, -b^2)}$
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*For $\cos(ax+by)$ replace “sin” by “cos”, and “cos” by “-sin” in z_p .

DIFFERENTIAL EQUATIONS

Differential equation	Method of solution
Separation of variables $f_1(x)g_1(y) dx + f_2(x)g_2(y) dy = 0$	$\int \frac{f_1(x)}{f_2(x)} dx + \int \frac{g_2(y)}{g_1(y)} dy = c$
Exact equation $M(x, y) dx + N(x, y) dy = 0$ where $\partial M / \partial y = \partial N / \partial x$	$\int M \partial x + \int \left(n - \frac{\partial}{\partial y} \int M \partial x \right) dy = c$ where ∂x indicates that the integration is to be performed with respect to x keeping y constant.
Linear first order equation $\frac{dy}{dx} + P(x)y = Q(x)$	$ye^{\int P dx} = \int Qe^{\int P dx} dx + c$
Bernoulli's equation $\frac{dy}{dx} + P(x)y = Q(x)y^n$	$ve^{(1-n)\int P dx} = (1-n) \int Qe^{(1-n)\int P dx} dx + c$ where $v = y^{1-n}$. If $n = 1$, the solution is $\ln y = \int (Q - P) dx + c$
Homogeneous equation $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$	$\ln x = \int \frac{dv}{F(v) - v} + c$ where $v = y/x$. If $F(v) = v$, the solution is $y = cx$
Reducible to homogeneous $(a_1x + b_1y + c_1) dx + (a_2x + b_2y + c_2) dy = 0$ $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Set $u = a_1x + b_1y + c_1$ $v = a_2x + b_2y + c_2$ Eliminate x and y and the equation becomes homogenous
Reducible to separable $(a_1x + b_1y + c_1) dx + (a_2x + b_2y + c_2) dy = 0$ $\frac{a_1}{a_2} = \frac{b_1}{b_2}$	Set $u = a_1x + b_1y$ Eliminate x or y and equation becomes separable

DIFFERENTIAL EQUATIONS (Continued)

$yF(xy) dx + xG(xy) dy = 0$	$\ln x = \int \frac{G(v) dv}{v\{G(v) - F(v)\}} + c$ where $v = xy$. If $G(v) = F(v)$, the solution is $xy = c$.
Linear, homogeneous second order equation $\frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$ b, c are real constants	Let m_1, m_2 be the roots of $m^2 + bm + c = 0$. Then there are 3 cases: Case 1. m_1, m_2 real and distinct: $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ Case 2. m_1, m_2 real and equal: $y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$ Case 3. $m_1 = p + qi, m_2 = p - qi$: $y = e^{px} (c_1 \cos qx + c_2 \sin qx)$ where $p = -b/2, q = \sqrt{4c - b^2}/2$
Linear, nonhomogeneous second order equation $\frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = R(x)$ b, c are real constants	There are 3 cases corresponding to those immediately above: Case 1. $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \frac{e^{m_1 x}}{m_1 - m_2} \int e^{-m_1 x} R(x) dx + \frac{e^{m_2 x}}{m_2 - m_1} \int e^{-m_2 x} R(x) dx$ Case 2. $y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + x e^{m_1 x} \int e^{-m_1 x} R(x) dx - e^{m_1 x} \int x e^{-m_1 x} R(x) dx$ Case 3. $y = e^{px} (c_1 \cos qx + c_2 \sin qx) + \frac{e^{px} \sin qx}{q} \int e^{-px} R(x) \cos qx dx - \frac{e^{px} \cos qx}{q} \int e^{-px} R(x) \sin qx dx$

DIFFERENTIAL EQUATIONS (Continued)

Euler or Cauchy equation $x^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = S(x)$	Putting $x = e^t$, the equation becomes $\frac{d^2 y}{dt^2} + (b - 1) \frac{dy}{dt} + cy = S(e^t)$ and can then be solved as a linear second order equation.
Bessel's equation $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (\lambda^2 x^2 - n^2)y = 0$	$y = c_1 J_n(\lambda x) + c_2 Y_n(\lambda x)$
Transformed Bessel's equation $x^2 \frac{d^2 y}{dx^2} + (2p + 1)x \frac{dy}{dx} + (\alpha^2 x^{2r} + \beta^2)y = 0$	$y = x^{-p} \left\{ c_1 J_{q/r} \left(\frac{\alpha}{r} x^r \right) + c_2 Y_{q/r} \left(\frac{\alpha}{r} x^r \right) \right\}$ where $q = \sqrt{p^2 - \beta^2}$.
Legendre's equation $(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n + 1)y = 0$	$y = c_1 P_n(x) + c_2 Q_n(x)$

FOURIER SERIES

If $f(x)$ is a bounded periodic function of period $2L$ (i.e. $f(x+2L)=f(x)$), and satisfies the *Dirichlet conditions*:

- A. In any period $f(x)$ is continuous, except possibly for a finite number of jump discontinuities.
- B. In any period $f(x)$ has only a finite number of maxima and minima.

Then $f(x)$ may be represented by the *Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where a_n and b_n are as determined below. This series will converge to $f(x)$ at every point where $f(x)$ is continuous, and to

$$\frac{f(x^+) + f(x^-)}{2}$$

(i.e., the average of the left-hand and right-hand limits) at every point where $f(x)$ has a jump discontinuity.

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, 3, \dots,$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots$$

we may also write

$$a_n = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \cos \frac{n\pi x}{L} dx \text{ and } b_n = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \sin \frac{n\pi x}{L} dx$$

where α is any real number. Thus if $\alpha = 0$,

$$a_n = \frac{1}{L} \int_0^{2L} f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, 3, \dots,$$

$$b_n = \frac{1}{L} \int_0^{2L} f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots$$

2. If in addition to the above restrictions, $f(x)$, is even (i.e., $f(-x) = f(x)$) the Fourier series reduces to

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

That is, $b_n = 0$. In this case, a simpler formula for a_n is

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, 3, \dots$$

3. If in addition to the restrictions in (1), $f(x)$ is an odd function (i.e., $f(-x) = -f(x)$), then the Fourier series reduces to

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

That is, $a_n = 0$. In this case, simpler formula for the b_n is

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots$$

4. If in addition to the restrictions in (2) above, $f(x) = -f(L - x)$, then a_n will be 0 for all even values of n , including $n = 0$. Thus in this case, the expansion reduces to

$$\sum_{m=1}^{\infty} a_{2m-1} \cos \frac{(2m-1)\pi x}{L}$$

5. If in addition to the restrictions in (3) above, $f(x) = f(L - x)$, then b_n will be 0 for all even values of n . Thus in this case, the expansion reduces to

$$\sum_{m=1}^{\infty} b_{2m-1} \sin \frac{(2m-1)\pi x}{L}$$

(The series in (4) and (5) are known as *odd-harmonic series*, since only the odd harmonics appear. Similar rules may be stated for even-harmonic series, but when a series appears in the even-harmonic form, it means that $2L$ has not been taken as the smallest period of $f(x)$. Since any integral multiple of a period is also a period, series obtained in this way will also work, but in general computation is simplified if $2L$ is taken to be the smallest period.)

6. If we write the Euler definitions for $\cos \theta$ and $\sin \theta$, we obtain the complex form of the Fourier Series known either as the “Complex Fourier Series” or the “Exponential Fourier Series” of $f(x)$. It is represented as

$$f(x) = \frac{1}{2} \sum_{n=-\infty}^{n=+\infty} c_n e^{i\omega_n x}$$

where

$$c_n = \frac{1}{L} \int_{-L}^L f(x) e^{-i\omega_n x} dx, \quad n = 0, \pm 1, \pm 2, \pm 3, \dots$$

with $\omega_n = \frac{n\pi}{L}$, $n = 0, \pm 1, \pm 2, \dots$

The set of coefficients $\{c_n\}$ is often referred to as the Fourier spectrum.

7. If both sine and cosine terms are present and if $f(x)$ is of period $2L$ and expandable by a Fourier series, it can be represented as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \sin \left(\frac{n\pi x}{L} + \phi_n \right), \quad \text{where } a_n = c_n \sin \phi_n, \\ b_n = c_n \cos \phi_n, \quad c_n = \sqrt{a_n^2 + b_n^2}, \quad \phi_n = \arctan \left(\frac{a_n}{b_n} \right)$$

It can also be represented as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos \left(\frac{n\pi x}{L} + \phi_n \right), \quad \text{where } a_n = c_n \cos \phi_n, \\ b_n = -c_n \sin \phi_n, \quad c_n = \sqrt{a_n^2 + b_n^2}, \quad \phi_n = \arctan \left(-\frac{b_n}{a_n} \right)$$

where ϕ_n is chosen so as to make a_n , b_n , and c_n hold.

8. The following table of trigonometric identities should be helpful for developing Fourier Series.

	n	n even	n odd	$n/2$ odd	$n/2$ even
$\sin n\pi$	0	0	0	0	0
$\cos n\pi$	$(-1)^n$	+1	-1	+1	+1
$^*\sin \frac{n\pi}{2}$		0	$(-1)^{(n-1)/2}$	0	0
$^*\cos \frac{n\pi}{2}$		$(-1)^{n/2}$	0	-1	+1
$\sin \frac{n\pi}{4}$			$\frac{\sqrt{2}}{2}(-1)^{(n^2+4n+11)/8}$	$(-1)^{(n-2)/4}$	0

*A useful formula for $\sin \frac{n\pi}{2}$ and $\cos \frac{n\pi}{2}$ is given by

$$\sin \frac{n\pi}{2} = \frac{(i)^{n+1}}{2} [(-1)^n - 1] \text{ and } \cos \frac{n\pi}{2} = \frac{(i)^n}{2} [(-1)^n + 1], \quad \text{where } i^2 = -1.$$

AUXILIARY FORMULAS FOR FOURIER SERIES

$$1 = \frac{4}{\pi} \left[\sin \frac{\pi x}{k} + \frac{1}{3} \sin \frac{3\pi x}{k} + \frac{1}{5} \sin \frac{5\pi x}{k} + \dots \right] \quad [0 < x < k]$$

$$x = \frac{2k}{\pi} \left[\sin \frac{\pi x}{k} - \frac{1}{2} \sin \frac{2\pi x}{k} + \frac{1}{3} \sin \frac{3\pi x}{k} - \dots \right] \quad [-k < x < k]$$

$$x = \frac{k}{2} - \frac{4k}{\pi^2} \left[\cos \frac{\pi x}{k} + \frac{1}{3^2} \cos \frac{3\pi x}{k} + \frac{1}{5^2} \cos \frac{5\pi x}{k} + \dots \right] \quad [0 < x < k]$$

$$x^2 = \frac{2k^2}{\pi^3} \left[\left(\frac{\pi^2}{1} - \frac{4}{1} \right) \sin \frac{\pi x}{k} - \frac{\pi^2}{2} \sin \frac{2\pi x}{k} + \left(\frac{\pi^2}{3} - \frac{4}{3^3} \right) \sin \frac{3\pi x}{k} - \frac{\pi^2}{4} \sin \frac{4\pi x}{k} + \left(\frac{\pi^2}{5} - \frac{4}{5^3} \right) \sin \frac{5\pi x}{k} + \dots \right] \quad [0 < x < k]$$

$$x^2 = \frac{k^2}{3} - \frac{4k^2}{\pi^2} \left[\cos \frac{\pi x}{k} - \frac{1}{2^2} \cos \frac{2\pi x}{k} + \frac{1}{3^2} \cos \frac{3\pi x}{k} - \frac{1}{4^2} \cos \frac{4\pi x}{k} + \dots \right] \quad [-k < x < k]$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

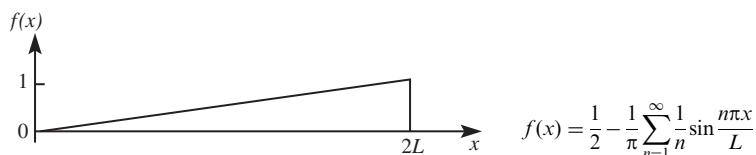
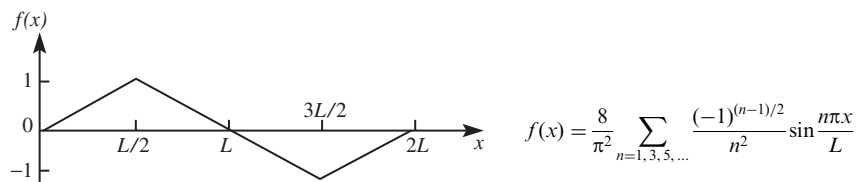
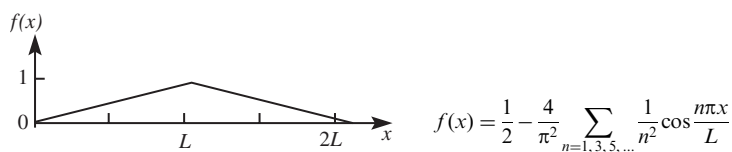
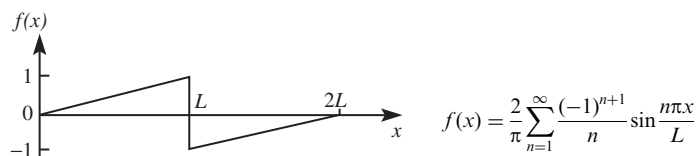
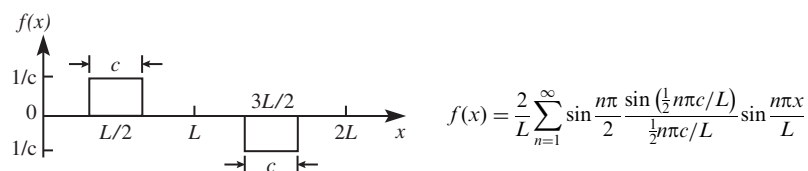
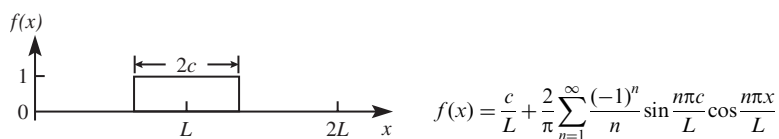
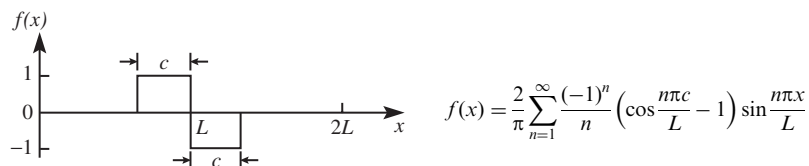
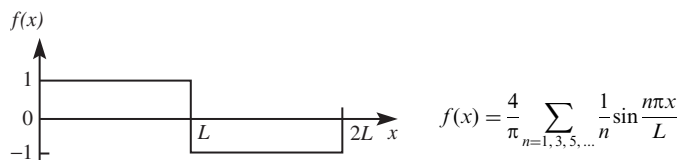
$$1 - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$$

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$$

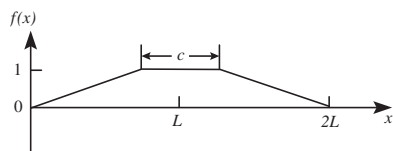
$$1 + \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$$

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \dots = \frac{\pi^2}{24}$$

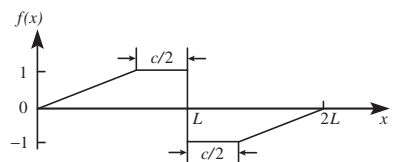
FOURIER EXPANSIONS FOR BASIC PERIODIC FUNCTIONS



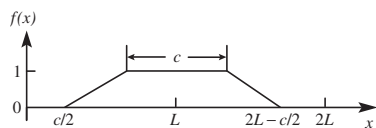
FOURIER EXPANSIONS FOR BASIC PERIODIC FUNCTIONS (Continued)



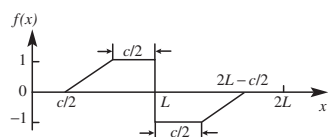
$$f(x) = \frac{1}{2}(1+a) + \frac{2}{\pi^2(1-a)} \sum_{n=1}^{\infty} \frac{1}{n^2} [(-1)^n \cos n\pi a - 1] \cos \frac{n\pi x}{L}; \quad \left(a = \frac{c}{2L}\right)$$



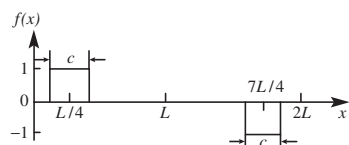
$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left[1 + \frac{\sin n\pi a}{n\pi(1-a)} \right] \sin \frac{n\pi x}{L}; \quad \left(a = \frac{c}{2L}\right)$$



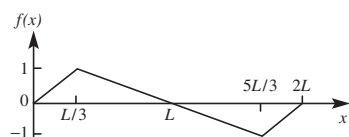
$$f(x) = \frac{1}{2} - \frac{4}{\pi^2(1-2a)} \sum_{n=1,3,5,\dots} \frac{1}{n^2} \cos n\pi a \cos \frac{n\pi x}{L}; \quad \left(a = \frac{c}{2L}\right)$$



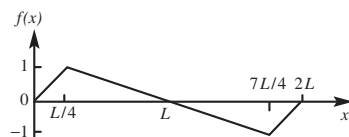
$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left[1 + \frac{1+(-1)^n}{n\pi(1-2a)} \sin n\pi a \right] \sin \frac{n\pi x}{L}; \quad \left(a = \frac{c}{2L}\right)$$



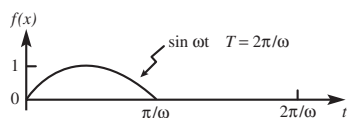
$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{4} \sin n\pi a \sin \frac{n\pi x}{L}; \quad \left(a = \frac{c}{2L}\right)$$



$$f(x) = \frac{9}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{3} \sin \frac{n\pi x}{L}; \quad \left(a = \frac{c}{2L}\right)$$



$$f(x) = \frac{32}{3\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{4} \sin \frac{n\pi x}{L}; \quad \left(a = \frac{c}{2L}\right)$$



$$f(x) = \frac{1}{\pi} + \frac{1}{2} \sin \omega t - \frac{2}{\pi} \sum_{n=2,4,6,\dots} \frac{1}{n^2-1} \cos n\omega t$$

Extracted from graphs and formulas, pages 372, 373, Differential Equations in Engineering Problems, Salvadori and Schwarz, published by Prentice-Hall, Inc., 1954.

THE FOURIER TRANSFORMS*

R. E. Gaskell

For a piecewise continuous function $F(x)$ over a finite interval $0 \leq x \leq \pi$, the *finite Fourier cosine transform* of $F(x)$ is

$$f_c(n) = \int_0^\pi F(x) \cos nx \, dx \quad (n = 0, 1, 2, \dots) \quad (1)$$

If x ranges over the interval $0 \leq x \leq L$, the substitution $x' = \pi x/L$ allows the use of this definition, also. The inverse transform is written.

$$\bar{F}(x) = \frac{1}{\pi} f_c(0) - \frac{2}{\pi} \sum_{n=1}^x f_c(n) \cos nx \quad (0 < x < \pi) \quad (2)$$

where $F(x) = \frac{[F(x+0) + F(x-0)]}{2}$. We observe that $F(x) = F(x)$ at point of continuity. The formula

$$\begin{aligned} f_c^{(2)}(n) &= \int_0^\pi F''(x) \cos nx \, dx \\ &= -n^2 f_c(n) - F'(0) + (-1)^n F'(\pi) \end{aligned} \quad (3)$$

makes the finite Fourier cosine transform useful in certain boundary value problems.

Analogously, the *finite Fourier sine transform* of $F(x)$ is

$$f_s(n) = \int_0^\pi F(x) \sin nx \, dx \quad (n = 1, 2, 3, \dots) \quad (4)$$

and

$$\bar{F}(x) = \frac{2}{\pi} \sum_{n=1}^\infty f_s(n) \sin nx \quad (0 < x < \pi) \quad (5)$$

Corresponding to (3) we have

$$\begin{aligned} f_s^{(2)}(n) &= \int_0^\pi F''(x) \sin nx \, dx \\ &= -n^2 f_s(n) - n F(0) - (-1)^n F(\pi) \end{aligned} \quad (6)$$

Fourier Transforms

If $F(x)$ is defined for $x \geq 0$ and is piecewise continuous over any finite interval, and if

$$\int_0^x F(x) \, dx$$

is absolutely convergent, then

$$f_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^x F(x) \cos(\alpha x) \, dx \quad (7)$$

is the *Fourier cosine transform* of $F(x)$. Furthermore,

$$\bar{F}(x) = \sqrt{\frac{2}{\pi}} \int_0^x f_c(\alpha) \cos(\alpha x) \, d\alpha \quad (8)$$

if $\lim_{x \rightarrow \infty} \frac{d^n F}{dx^n} = 0$, an important property of the Fourier cosine transform

$$\begin{aligned} f_c^{(2r)}(\alpha) &= \sqrt{\frac{2}{\pi}} \int_0^x \left(\frac{d^{2r} F}{dx^{2r}} \right) \cos(\alpha x) \, dx \\ &= -\sqrt{\frac{2}{\pi}} \sum_{n=0}^{r-1} (-1)^n a_{2r-2n-1} \alpha^{2n} + (-1)^r \alpha^{2r} f_c(\alpha) \end{aligned} \quad (9)$$

where $\lim_{x \rightarrow \infty} \frac{d^r F}{dx^r} = a_r$, makes it useful in the solution of many problems.

Under the same conditions.

$$f_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^x F(x) \sin(\alpha x) \, dx \quad (10)$$

*From Beyer, W. H., Ed., *CRC Handbook of Mathematical Sciences*, 5th ed., CRC Press, Boca Raton, 1978, 592–598. With permission.

defines the *Fourier sine transform of $F(x)$* , and

$$\bar{F}(x) = \sqrt{\frac{2}{\pi}} \int_0^x f_s(\alpha) \sin(\alpha x) d\alpha \quad (11)$$

Corresponding to (9) we have

$$\begin{aligned} f_s^{(2r)}(\alpha) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d^{2r}F}{dx^{2r}} \sin(\alpha x) dx \\ &= -\sqrt{\frac{2}{\pi}} \sum_{n=1}^r (-1)^n \alpha^{2n-1} a_{2r-2n} + (-1)^{r-1} \alpha^{2r} f_s(\alpha) \end{aligned} \quad (12)$$

Similarly, if $F(x)$ is defined for $-\infty < x < \infty$, and if $\int_{-\infty}^\infty F(x) dx$ is absolutely convergent, then

$$f(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty F(x) e^{i\alpha x} dx \quad (13)$$

is the *Fourier transform of $F(x)$* , and

$$\bar{F}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(\alpha) e^{-i\alpha x} d\alpha \quad (14)$$

Also, if

$$\lim_{|x| \rightarrow \infty} \left| \frac{d^n F}{dx^n} \right| = 0 \quad (n = 1, 2, \dots, r-1)$$

then

$$f^{(r)}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty F^{(r)}(x) e^{i\alpha x} dx = (-i\alpha)^r f(\alpha) \quad (15)$$

Finite Sine Transforms

$f_s(n)$	$F(x)$
1 $f_s(n) = \int_0^\pi F(x) \sin nx dx \quad (n = 1, 2, \dots)$	$F(x)$
2 $(-1)^{n+1} f_s(n)$	$F(\pi - x)$
3 $\frac{1}{n}$	$\frac{\pi - x}{\pi}$
4 $\frac{(-1)^{n+1}}{n}$	$\frac{x}{\pi}$
5 $\frac{1 - (-1)^n}{n}$	1
6 $\frac{2}{n^2} \sin \frac{n\pi}{2}$	$\begin{cases} x & \text{when } 0 < x < \pi/2 \\ \pi - x & \text{when } \pi/2 < x < \pi \end{cases}$
7 $\frac{(-1)^{n+1}}{n^3}$	$\frac{x(\pi^2 - x^2)}{6\pi}$
8 $\frac{1 - (-1)^n}{n^3}$	$\frac{x(\pi - x)}{2}$
9 $\frac{\pi^2(-1)^{n-1}}{n} - \frac{2[1 - (-1)^n]}{n^3}$	x^2
10 $\pi(-1)^n \left(\frac{6}{n^3} - \frac{\pi^2}{n} \right)$	x^3
11 $\frac{n}{n^2 + c^2} [1 - (-1)^n e^{c\pi}]$	e^{cx}
12 $\frac{n}{n^2 + c^2}$	$\frac{\sinh c(\pi - x)}{\sinh c\pi}$
13 $\frac{n}{n^2 - k^2} \quad (k \neq 0, 1, 2, \dots)$	$\frac{\sin k(\pi - x)}{\sin k\pi}$
14 $\begin{cases} \frac{\pi}{2} & \text{when } n = m \\ 0 & \text{when } n \neq m \end{cases} \quad (m = 1, 2, \dots)$	$\sin mx$

	$f_s(n)$	$F(x)$
15	$\frac{n}{n^2 - k^2} [1 - (-1)^n \cos k\pi]$ ($k \neq 1, 2, \dots$)	$\cos kx$
16	$\begin{cases} \frac{n}{n^2 - m^2} [1 - (-1)^{n+m}] \\ 0 & \text{when } n \neq m = 1, 2, \dots \\ & \text{when } n = m \end{cases}$	$\cos mx$
17	$\frac{n}{(n^2 - k^2)^2} (k \neq 0, 1, 2, \dots)$	$\frac{\pi \sin kx}{2k \sin^2 k\pi} - \frac{x \cos k(\pi - x)}{2k \sin k\pi}$
18	$\frac{b^n}{n} (b \leq 1)$	$\frac{2}{\pi} \arctan \frac{b \sin x}{1 - b \cos x}$
19	$\frac{1 - (-1)^n}{n} b^n \quad (b \leq 1)$	$\frac{2}{\pi} \arctan \frac{2b \sin x}{1 - b^2}$

Finite Cosine Transforms

	$f_c(n)$	$F(x)$
1	$f_c(n) = \int_0^\pi F(x) \cos nx \, dx \quad (n = 0, 1, 2, \dots)$	$F(x)$
2	$(-1)^n f_c(n)$	$F(\pi - x)$
3	0 when $n = 1, 2, \dots; f_c(0) = \pi$	1
4	$\frac{2}{n} \sin \frac{n\pi}{2}; f_c(0) = 0$	$\begin{cases} 1 & \text{when } 0 < x < \pi/2 \\ -1 & \text{when } \pi/2 < x < \pi \end{cases}$
5	$-\frac{1 - (-1)^n}{n^2}; f_c(0) = \frac{\pi^2}{2}$	x
6	$\frac{(-1)^n}{n^2}; f_c(0) = \frac{\pi^2}{6}$	$\frac{x^2}{2\pi}$
7	$\frac{1}{n^2}; f_c(0) = 0$	$\frac{(\pi - x)^2}{2\pi} - \frac{\pi}{6}$
8	$3\pi^2 \frac{(-1)^n}{n^2} - 6 \frac{1 - (-1)^n}{n^4}; f_c(0) = \frac{\pi^4}{4}$	x^3
9	$\frac{(-1)^n e^c \pi - 1}{n^2 + c^2}$	$\frac{1}{c} e^{cx}$
10	$\frac{1}{n^2 + c^2}$	$\frac{\cosh c(\pi - x)}{c \sinh c\pi}$
11	$\frac{k}{n^2 - k^2} [(-1)^n \cos \pi k - 1]$ ($k \neq 0, 1, 2, \dots$)	$\sin kx$
12	$\frac{(-1)^{n+m} - 1}{n^2 - m^2}; f_c(m) = 0 \quad (m = 1, 2, \dots)$	$\frac{1}{m} \sin mx$
13	$\frac{1}{n^2 - k^2} \quad (k \neq 0, 1, 2, \dots)$	$-\frac{\cos k(\pi - x)}{k \sin k\pi}$
14	0 when $n = 1, 2, \dots;$ $f_c(m) = \frac{\pi}{2} \quad (m = 1, 2, \dots)$	$\cos mx$

Fourier Sine Transforms*

	$F(x)$	$f_s(\alpha)$
1	$\begin{cases} 1 & (0 < x < a) \\ 0 & (x > a) \end{cases}$	$\sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos \alpha}{\alpha} \right]$
2	$x^{p-1} \quad (0 < p < 1)$	$\sqrt{\frac{2}{\pi}} \frac{\Gamma(p)}{\alpha^p} \sin \frac{p\pi}{2}$
3	$\begin{cases} \sin x & (0 < x < a) \\ 0 & (x > a) \end{cases}$	$\frac{1}{\sqrt{2\pi}} \left[\frac{\sin[a(1 - \alpha)]}{1 - \alpha} - \frac{\sin[a(1 + \alpha)]}{1 + \alpha} \right]$
4	e^{-x}	$\sqrt{\frac{2}{\pi}} \left[\frac{\alpha}{1 + \alpha^2} \right]$
5	$xe^{-x^2/2}$	$\alpha e^{-\alpha^2/2}$
6	$\cos \frac{x^2}{2}$	$\sqrt{2} \left[\sin \frac{\alpha^2}{2} c \left(\frac{\alpha^2}{2} \right) - \cos \frac{\alpha^2}{2} S \left(\frac{\alpha^2}{2} \right) \right]^*$

$F(x)$	$f_s(\alpha)$
7 $\sin \frac{x^2}{2}$	$\sqrt{2} \left[\cos \frac{\alpha^2}{2} C\left(\frac{\alpha^2}{2}\right) + \sin \frac{\alpha^2}{2} S\left(\frac{\alpha^2}{2}\right) \right]^*$

* $C(y)$ and $S(y)$ are the Fresnel integrals

$$C(y) = \frac{1}{\sqrt{2\pi}} \int_0^y \frac{1}{\sqrt{t}} \cos t \, dt,$$

$$S(y) = \frac{1}{\sqrt{2\pi}} \int_0^y \frac{1}{\sqrt{t}} \sin t \, dt$$

*More extensive tables of the Fourier sine and cosine transforms can be found in Fritz Oberhettinger, *Tabellen zur-Fourier Transformation*, Springer, 1957.

Fourier Cosine Transforms

$F(x)$	$f_c(\alpha)$
1 $\begin{cases} 1 & (0 < x < a) \\ 0 & (x > a) \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin a\alpha}{\alpha}$
2 $x^{p-1} \quad (0 < p < 1)$	$\sqrt{\frac{2}{\pi}} \frac{\Gamma(p)}{\alpha^p} \cos \frac{p\pi}{2}$
3 $\begin{cases} \cos x & (0 < x < a) \\ 0 & (x > a) \end{cases}$	$\frac{1}{\sqrt{2\pi}} \left[\frac{\sin[a(1-\alpha)]}{1-\alpha} + \frac{\sin[a(1+\alpha)]}{1+\alpha} \right]$
4 e^{-x}	$\sqrt{\frac{2}{\pi}} \left(\frac{1}{1+\alpha^2} \right)$
5 $e^{-x^2/2}$	$e^{-\alpha^2/2}$
6 $\cos \frac{x^2}{2}$	$\cos \left(\frac{\alpha^2}{2} - \frac{\pi}{4} \right)$
7 $\sin \frac{x^2}{2}$	$\cos \left(\frac{\alpha^2}{2} + \frac{\pi}{4} \right)$

Fourier Transforms

$F(x)$	$f_s(\alpha)$
1 $\frac{\sin ax}{x}$	$\begin{cases} \sqrt{\frac{\pi}{2}} & \alpha < a \\ 0 & \alpha > a \end{cases}$
2 $\begin{cases} e^{iwx} & (p, x < q) \\ 0 & (x < p, x > q) \end{cases}$	$\frac{i}{\sqrt{2\pi}} \frac{e^{ip(w+\alpha)} - e^{iq(w+\alpha)}}{(w+\alpha)}$
3 $\begin{cases} e^{-cx+ix} & (x > 0) \\ 0 & (x < 0) \end{cases} \quad (c > 0)$	$\frac{i}{\sqrt{2\pi}(w+\alpha+ic)}$
4 $e^{-px^2} \quad R(p) > 0$	$\frac{1}{\sqrt{2p}} e^{-\alpha^2/4p}$
5 $\cos px^2$	$\frac{1}{\sqrt{2p}} \cos \left[\frac{\alpha^2}{4p} - \frac{\pi}{4} \right]$
6 $\sin px^2$	$\frac{1}{\sqrt{2p}} \cos \left[\frac{\alpha^2}{4p} + \frac{\pi}{4} \right]$
7 $ x ^{-p} \quad (0 < p < 1)$	$\sqrt{\frac{2}{\pi}} \frac{\Gamma(1-p) \sin \frac{p\pi}{2}}{ \alpha ^{(1-p)}}$
8 $\frac{e^{-a x }}{\sqrt{ x }}$	$\frac{\sqrt{\sqrt{(a^2+\alpha^2)}+a}}{\sqrt{a^2+\alpha^2}}$
9 $\frac{\cosh ax}{\cosh \pi x} \quad (-\pi < a < \pi)$	$\sqrt{\frac{2}{\pi}} \frac{\cos \frac{a}{2} \cosh \frac{a}{2}}{\cosh \alpha + \cos a}$
10 $\frac{\sinh ax}{\sinh \pi x} \quad (-\pi < a < \pi)$	$\frac{1}{\sqrt{2\pi}} \frac{\sin a}{\cosh \alpha + \cos a}$
11 $\begin{cases} \frac{1}{\sqrt{a^2-x^2}} & (x < a) \\ 0 & (x > a) \end{cases}$	$\sqrt{\frac{\pi}{2}} J_0(a\alpha)$

	$F(x)$	$f(\alpha)$
12	$\frac{\sin[b\sqrt{a^2+x^2}]}{\sqrt{a^2+x^2}}$	$\begin{cases} 0 & (\alpha > b) \\ \sqrt{\frac{\pi}{2}} J_0(a\sqrt{b^2-\alpha^2}) & (\alpha < b) \end{cases}$
13	$\begin{cases} p_a(x) & (x < 1) \\ 0 & (x > 1) \end{cases}$	$\frac{i^n}{\sqrt{\alpha}} J_{n+\frac{1}{2}}(\alpha)$
14	$\begin{cases} \frac{\cos[b\sqrt{a^2-x^2}]}{\sqrt{a^2-x^2}} & (x < a) \\ 0 & (x > a) \end{cases}$	$\sqrt{\frac{\pi}{2}} J_0(a\sqrt{a^2+b^2})$
15	$\begin{cases} \frac{\cosh[b\sqrt{a^2-x^2}]}{\sqrt{a^2-x^2}} & (x < a) \\ 0 & (x > a) \end{cases}$	$\sqrt{\frac{\pi}{2}} J_0(a\sqrt{a^2-b^2})$

* More extensive tables of Fourier transforms can be found in W. Magnus and F. Oberhettinger, *Formulas and Theorems of the Special Functions of Mathematical Physics*. Chelsea, 1949, 116–120.

The following functions appear among the entries of the tables on transforms.

Function	Definition	Name
$Ei(x)$	$\int_{-x}^x \frac{e^v}{v} dv$; or sometimes defined as $-Ei(-x) = \int_x^\infty \frac{e^{-v}}{v} dv$	Sine, Cosine, and Exponential Integral tables pages 548–556
$Si(x)$	$\int_0^x \frac{\sin v}{v} dv$	Sine, Cosine, and Exponential Integral tables pages 548–556
$Ci(x)$	$\int_x^\infty \frac{\cos v}{v} dv$; or sometimes defined as negative of this integral	Sine, Cosine, and Exponential Integral tables pages 548–556
$erf(x)$	$\frac{2}{\sqrt{\pi}} \int_0^x e^{-v^2} dv$	Error function
$erfc(x)$	$1 - erf(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-v^2} dv$	Complementary function to error function
$L_n(x)$	$\frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x})$, $n = 0, 1, \dots$	Laguerre polynomial of degree n

SERIES EXPANSION

The expression in parentheses following certain of the series indicates the region of convergence. If not otherwise indicated it is to be understood that the series converges for all finite values of x .

BINOMIAL

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \dots \quad (y^2 < x^2)$$

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)x^2}{2!} \pm \frac{n(n-1)(n-2)x^3}{3!} + \dots \quad (x^2 < 1)$$

$$(1 \pm x)^{-n} = 1 \mp nx + \frac{n(n+1)x^2}{2!} \mp \frac{n(n+1)(n+2)x^3}{3!} + \dots \quad (x^2 < 1)$$

$$(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \dots \quad (x^2 < 1)$$

$$(1 \pm x)^{-2} = 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp 6x^5 + \dots \quad (x^2 < 1)$$

REVERSION OF SERIES

Let a series be represented by

$$y = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots \quad (a_j \neq 0)$$

to find the coefficients of the series

$$x = A_1y + A_2y^2 + A_3y^3 + A_4y^4 + \dots$$

$$A_1 = \frac{1}{a_1} \quad A_2 = -\frac{a_2}{a_1^2} \quad A_3 = \frac{1}{a_1^3}(2a_2^2 - a_1a_3)$$

$$A_4 = \frac{1}{a_1^4}(5a_1a_2a_3 - a_1^2a_4 - 5a_2^3)$$

SERIES EXPANSION

The expression in parentheses following certain of the series indicates the region of convergence. If not otherwise indicated it is to be understood that the series converges for all finite values of x .

BINOMIAL

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \dots \quad (y^2 < x^2)$$

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)x^2}{2!} \pm \frac{n(n-1)(n-2)x^3}{3!} + \dots \quad (x^2 < 1)$$

$$(1 \pm x)^{-n} = 1 \mp nx + \frac{n(n+1)x^2}{2!} \mp \frac{n(n+1)(n+2)x^3}{3!} + \dots \quad (x^2 < 1)$$

$$(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \dots \quad (x^2 < 1)$$

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REVERSION OF SERIES

Let a series be represented by

$$y = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots \quad (a_j \neq 0)$$

to find the coefficients of the series

$$x = A_1y + A_2y^2 + A_3y^3 + A_4y^4 + \dots$$

$$A_1 = \frac{1}{a_1} \quad A_2 = -\frac{a_2}{a_1^3} \quad A_3 = \frac{1}{a_1^5}(2a_2^2 - a_1a_3)$$

$$A_4 = \frac{1}{a_1^7}(5a_1a_2a_3 - a_1^2a_4 - 5a_2^3)$$

$$\begin{aligned}
A_5 &= \frac{1}{a_1^9} (6a_1^2 a_2 a_4 + 3a_1^2 a_3^2 + 14a_2^4 - a_1^3 a_5 - 21a_1 a_2^2 a_3) \\
A_6 &= \frac{1}{a_1^{11}} (7a_1^3 a_2 a_5 + 7a_1^3 a_3 a_4 + 84a_1 a_2^3 a_3 - a_1^4 a_6 - 28a_1^2 a_2^2 a_4 - 28a_1^2 a_2 a_3^2 - 42a_2^5) \\
A_7 &= \frac{1}{a_1^{13}} (8a_1^4 a_2 a_6 + 8a_1^4 a_3 a_5 + 4a_1^4 a_4^2 + 120a_1^2 a_2^3 a_4 + 180a_1^2 a_2^2 a_3^2 + 132a_2^6 - a_1^5 a_7 \\
&\quad - 36a_1^3 a_2^2 a_5 - 72a_1^3 a_2 a_3 a_4 - 12a_1^3 a_3^3 - 330a_1 a_2^4 a_3)
\end{aligned}$$

TAYLOR

1. $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \dots$ (Taylor's Series)
(Increment form)
2. $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$
 $= f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \dots$
3. If $f(x)$ is a function possessing derivatives of all orders throughout the interval $a \leq x \leq b$, then there is a value X , with aXb , such that

$$\begin{aligned}
f(b) &= f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \dots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-a)^n}{n!}f^{(n)}(X) \\
f(a+h) &= f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h), \quad b = a+h, \quad 0 < \theta < 1.
\end{aligned}$$

or

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots + (x-a)^{n-1}\frac{f^{(n-1)}(a)}{(n-1)!} + R_n,$$

where

$$R_n = \frac{f^{(n)}[a + \theta \cdot (x-a)]}{n!}(x-a)^n, \quad 0 < \theta < 1.$$

The above forms are known as Taylor's series with the remainder term.

4. Taylor's series for a function of two variables

$$\begin{aligned}
\text{If } \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right) f(x, y) &= h \frac{\partial f(x, y)}{\partial x} + k \frac{\partial f(x, y)}{\partial y}; \\
\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^2 f(x, y) &= h^2 \frac{\partial^2 f(x, y)}{\partial x^2} + 2hk \frac{\partial^2 f(x, y)}{\partial x \partial y} + k^2 \frac{\partial^2 f(x, y)}{\partial y^2}
\end{aligned}$$

etc., and if $\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^n f(x, y) \Big|_{x=a, y=b}$ with the bar and subscripts means that after differentiation we are to replace x by a and y by b ,

$$f(a+h, b+k) = f(a, b) + \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right) f(x, y) \Big|_{x=a, y=b} + \dots + \frac{1}{n!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^n f(x, y) \Big|_{x=a, y=b} + \dots$$

MACLAURIN

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots + x^{n-1}\frac{f^{(n-1)}(0)}{(n-1)!} + R_n,$$

where

$$R_n = \frac{x^n f^{(n)}(\theta x)}{n!}, \quad 0 < \theta < 1.$$

EXPONENTIAL

$$\begin{aligned}
e &= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots
\end{aligned}$$

(all real values of x)

$$a^x = 1 + x \log_e a + \frac{(x \log_e a)^2}{2!} + \frac{(x \log_e a)^3}{3!} + \dots$$

$$e^x = e^a \left[1 + (x-a) + \frac{(x-a)^2}{2!} + \frac{(x-a)^3}{3!} + \dots \right]$$

LOGARITHMIC

$$\log_e x = \frac{x-1}{x} + \frac{1}{2} \left(\frac{x-1}{x} \right)^2 + \frac{1}{3} \left(\frac{x-1}{x} \right)^3 + \dots \quad (x > \tfrac{1}{2})$$

$$\log_e x = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \dots \quad (2 \geq x > 0)$$

$$\log_e x = 2 \left[\frac{x-1}{x+1} + \frac{1}{3} \left(\frac{x-1}{x+1} \right)^3 + \frac{1}{5} \left(\frac{x-1}{x+1} \right)^5 + \dots \right] \quad (x > 0)$$

$$\log_e(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \dots \quad (-1 < x \leq 1)$$

$$\log_e(n+1) - \log_e(n-1) = 2 \left[\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \dots \right]$$

$$\log_e(a+x) = \log_e a + 2 \left[\frac{x}{2a+x} + \frac{1}{3} \left(\frac{x}{2a+x} \right)^3 + \frac{1}{5} \left(\frac{x}{2a+x} \right)^5 + \dots \right] \quad (a > 0, -a < x < +\infty)$$

$$\log_e \frac{1+x}{1-x} = 2 \left[x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2n-1}}{2n-1} + \dots \right] \quad -1 < x < 1$$

$$\log_e x = \log_e a + \frac{(x-a)}{a} - \frac{(x-a)^2}{2a^2} + \frac{(x-a)^3}{3a^3} - \dots \quad 0 < x \leq 2a$$

TRIGONOMETRIC

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad (\text{all real values of } x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad (\text{all real values of } x)$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \dots + \frac{(-1)^{n-1} 2^{2n} (2^{2n} - 1) B_{2n}}{(2n)!} x^{2n-1} + \dots,$$

$$\left[x^2 < \frac{\pi^2}{4}, \text{ and } B_n \text{ represents the } n\text{th Bernoulli number.} \right]$$

$$\cot x = \frac{1}{x} - \frac{x}{3} + \frac{x^3}{45} - \frac{2x^5}{945} + \frac{x^7}{4725} - \dots - \frac{(-1)^{n+1} 2^{2n}}{(2n)!} B_{2n} x^{2n-1} - \dots,$$

$$[x^2 < \pi^2, \text{ and } B_n \text{ represents the } n\text{th Bernoulli number.}]$$

$$\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \dots + \frac{(-1)^n E_{2n} x^{2n}}{(2n)!} + \dots,$$

$$\left[x^2 < \frac{\pi^2}{4}, \text{ and } E_n \text{ represents the } n\text{th Euler number.} \right]$$

$$\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7}{360}x^3 + \frac{31}{15,120}x^5 + \frac{127}{604,800}x^7 + \dots$$

$$+ \frac{(-1)^{n+1} 2(2^{2n-1} - 1) B_{2n} x^{2n-1}}{(2n)!} + \dots,$$

$$[x^2 < \pi^2, \text{ and } B_n \text{ represents } n\text{th Bernoulli number.}]$$

$$\sin x = x \left(1 - \frac{x^2}{\pi^2} \right) \left(1 - \frac{x^2}{2^2 \pi^2} \right) \left(1 - \frac{x^2}{3^2 \pi^2} \right) \dots \quad (x^2 < \infty)$$

$$\cos x = \left(1 - \frac{4x^2}{\pi^2} \right) \left(1 - \frac{4x^2}{3^2 \pi^2} \right) \left(1 - \frac{4x^2}{5^2 \pi^2} \right) \dots \quad (x^2 < \infty)$$

$$\sin^{-1} x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} x^7 + \dots \quad \left(x^2 < 1, -\frac{\pi}{2} < \sin^{-1} x < \frac{\pi}{2} \right)$$

$$\cos^{-1} x = \frac{\pi}{2} - \left(x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} x^7 + \dots \right) \quad (x^2 < 1, 0 < \cos^{-1} x < \pi)$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad (x^2 < 1)$$

$$\tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \dots \quad (x > 1)$$

$$\tan^{-1} x = -\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \dots \quad (x < -1)$$

$$\cot^{-1} x = \frac{\pi}{2} - x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \dots \quad (x^2 < 1)$$

$$\log_e \sin x = \log_e x - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \dots \quad (x^2 < \pi^2)$$

$$\log_e \cos x = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \dots \quad \left(x^2 < \frac{\pi^2}{4}\right)$$

$$\log_e \tan x = \log_e x + \frac{x^2}{3} + \frac{7x^4}{90} + \frac{62x^6}{2835} + \dots \quad \left(x^2 < \frac{\pi^2}{4}\right)$$

$$e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^5}{5!} - \frac{3x^6}{6!} + \frac{56x^7}{7!} + \dots$$

$$e^{\cos x} = e \left(1 - \frac{x^2}{2!} + \frac{4x^4}{4!} - \frac{31x^6}{6!} + \dots \right)$$

$$e^{\tan x} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + \frac{37x^5}{5!} + \dots \quad \left(x^2 < \frac{\pi^2}{4}\right)$$

$$\begin{aligned} \sin x &= \sin a + (x - a) \cos a - \frac{(x - a)^2}{2!} \sin a \\ &\quad - \frac{(x - a)^3}{3!} \cos a + \frac{(x - a)^4}{4!} \sin a + \dots \end{aligned}$$

VECTOR ANALYSIS

Definitions

Any quantity which is completely determined by its magnitude is called a *scalar*. Examples of such are mass, density, temperature, etc. Any quantity which is completely determined by its magnitude and direction is called a *vector*. Examples of such are velocity, acceleration, force, etc. A vector quantity is represented by a directed line segment, the length of which represents the magnitude of the vector. A vector quantity is usually represented by a boldfaced letter such as $\mathbf{V}$. Two vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are equal to one another if they have equal magnitudes and are acting in the same directions. A negative vectors written as $-\mathbf{V}$, is one which acts in the opposite direction to $\mathbf{V}$, but is of equal magnitude to it. If we represent the magnitude of $\mathbf{V}$ by v , we write $|\mathbf{V}| = v$. A vector parallel to $\mathbf{V}$, but equal to the reciprocal of its magnitude is written as $\mathbf{V}^{-1}$ or as $\frac{1}{\mathbf{V}}$.

The *unit vector* $\frac{\mathbf{V}}{v}$ ($v \neq 0$) is that vector which has the same direction as $\mathbf{V}$, but has a magnitude of unity (sometimes represented as $\mathbf{V}_0$ or $\hat{\mathbf{v}}$).

Vector Algebra

The vector sum of $\mathbf{V}_1$ and $\mathbf{V}_2$ is represented by $\mathbf{V}_1 + \mathbf{V}_2$. The vector sum of $\mathbf{V}_1$ and $-\mathbf{V}_2$, or the difference of the vector $\mathbf{V}_2$ from $\mathbf{V}_1$ is represented by $\mathbf{V}_1 - \mathbf{V}_2$.

If r is a scalar, then $r\mathbf{V} = \mathbf{V}r$, and represents a vector r times the magnitude of $\mathbf{V}$, in the same direction as $\mathbf{V}$ if r is positive, and in the opposite direction if r is negative. If r and s are scalars, $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$, vectors, then the following rules of scalars and vectors hold:

$$\begin{aligned}\mathbf{V}_1 + \mathbf{V}_2 &= \mathbf{V}_2 + \mathbf{V}_1 \\ (r + s)\mathbf{V}_1 &= r\mathbf{V}_1 + s\mathbf{V}_1; \quad r(\mathbf{V}_1 + \mathbf{V}_2) = r\mathbf{V}_1 + r\mathbf{V}_2 \\ \mathbf{V}_1 + (\mathbf{V}_2 + \mathbf{V}_3) &= (\mathbf{V}_1 + \mathbf{V}_2) + \mathbf{V}_3 = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3\end{aligned}$$

Vectors in Space

A plane is described by two distinct vectors $\mathbf{V}_1$ and $\mathbf{V}_2$. Should these vectors not intersect each other, then one is displaced parallel to itself until they do (fig. 1). Any other vector $\mathbf{V}$ lying in this plane is given by

$$\mathbf{V} = r\mathbf{V}_1 + s\mathbf{V}_2$$

A *position vector* specifies the position in space of a point relative to a fixed origin. If therefore $\mathbf{V}_1$ and $\mathbf{V}_2$ are the position vectors of the points A and B , relative to the origin O , then any point P on the line AB has a position vector $\mathbf{V}$ given by

$$\mathbf{V} = r\mathbf{V}_1 + (1 - r)\mathbf{V}_2$$

The scalar “ r ” can be taken as the parametric representation of P since $r = 0$ implies $P = B$ and $r = 1$ implies $P = A$ (fig. 2). If P divides the line AB in the ratio $r : s$ then

$$\mathbf{V} = \left(\frac{r}{r+s}\right)\mathbf{V}_1 + \left(\frac{s}{r+s}\right)\mathbf{V}_2$$

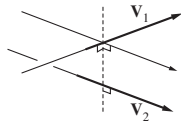


Figure 1.

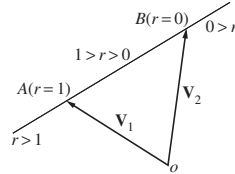


Figure 2.

The vectors $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \dots, \mathbf{V}_n$ are said to be *linearly dependent* if there exist scalars $r_1, r_2, r_3, \dots, r_n$, not all zero, such that

$$r_1 \mathbf{V}_1 + r_2 \mathbf{V}_2 + \dots + r_n \mathbf{V}_n = \mathbf{0}$$

A vector $\mathbf{V}$ is linearly dependent upon the set of vectors $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \dots, \mathbf{V}_n$ if

$$\mathbf{V} = r_1 \mathbf{V}_1 + r_2 \mathbf{V}_2 + r_3 \mathbf{V}_3 + \dots + r_n \mathbf{V}_n$$

Three vectors are linearly dependent if and only if they are co-planar.

All points in space can be uniquely determined by linear dependence upon three *base vectors* i.e., three vectors any one of which is linearly independent of the other two. The simplest set of base vectors are the unit vectors along the coordinate Ox, Oy and Oz axes. These are usually designated by $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ respectively.

If $\mathbf{V}$ is a vector in space, and a, b and c are the respective magnitudes of the projections of the vector along the axes then

$$\mathbf{V} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

and

$$v = \sqrt{a^2 + b^2 + c^2}$$

and the direction cosines of $\mathbf{V}$ are

$$\cos \alpha = a/v, \quad \cos \beta = b/v, \quad \cos \gamma = c/v.$$

The law of addition yields

$$\mathbf{V}_1 + \mathbf{V}_2 = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j} + (c_1 + c_2)\mathbf{k}$$

The Scalar, Dot, or Inner Product of Two Vectors $\mathbf{V}_1$ and $\mathbf{V}_2$

This product is represented as $\mathbf{V}_1 \cdot \mathbf{V}_2$ and is defined to be equal to $v_1 v_2 \cos \theta$, where θ is the angle from $\mathbf{V}_1$ to $\mathbf{V}_2$, i.e.,

$$\mathbf{V}_1 \cdot \mathbf{V}_2 = v_1 v_2 \cos \theta$$

The following rules apply for this product:

$$\mathbf{V}_1 \cdot \mathbf{V}_2 = a_1 a_2 + b_1 b_2 + c_1 c_2 = \mathbf{V}_2 \cdot \mathbf{V}_1$$

It should be noted that this verifies that scalar multiplication is commutative.

$$(\mathbf{V}_1 + \mathbf{V}_2) \cdot \mathbf{V}_3 = \mathbf{V}_1 \cdot \mathbf{V}_3 + \mathbf{V}_2 \cdot \mathbf{V}_3$$

$$\mathbf{V}_1 \cdot (\mathbf{V}_2 + \mathbf{V}_3) = \mathbf{V}_1 \cdot \mathbf{V}_2 + \mathbf{V}_1 \cdot \mathbf{V}_3$$

If $\mathbf{V}_1$ is perpendicular to $\mathbf{V}_2$ then $\mathbf{V}_1 \cdot \mathbf{V}_2 = 0$, and if $\mathbf{V}_1$ is parallel to $\mathbf{V}_2$ then $\mathbf{V}_1 \cdot \mathbf{V}_2 = v_1 v_2 = r w_1^2$

In particular

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1,$$

and

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$$

The Vector or Cross Product of Vectors $\mathbf{V}_1$ and $\mathbf{V}_2$

This product is represented as $\mathbf{V}_1 \times \mathbf{V}_2$ and is defined to be equal to $v_1 v_2 (\sin \theta) \mathbf{1}$, where θ is the angle from $\mathbf{V}_1$ to $\mathbf{V}_2$ and $\mathbf{1}$ is a unit vector perpendicular to the plane of $\mathbf{V}_1$ and $\mathbf{V}_2$, and so directed that a right-handed screw driven in the direction of $\mathbf{1}$ would carry $\mathbf{V}_1$ into $\mathbf{V}_2$, i.e.,

$$\mathbf{V}_1 \times \mathbf{V}_2 = v_1 v_2 (\sin \theta) \mathbf{1}$$

and

$$\tan \theta = \frac{|\mathbf{V}_1 \times \mathbf{V}_2|}{\mathbf{V}_1 \cdot \mathbf{V}_2}$$

The following rules apply for vector products:

$$\mathbf{V}_1 \times \mathbf{V}_2 = -\mathbf{V}_2 \times \mathbf{V}_1$$

$$\mathbf{V}_1 \times (\mathbf{V}_2 + \mathbf{V}_3) = \mathbf{V}_1 \times \mathbf{V}_2 + \mathbf{V}_1 \times \mathbf{V}_3$$

$$(\mathbf{V}_1 + \mathbf{V}_2) \times \mathbf{V}_3 = \mathbf{V}_1 \times \mathbf{V}_3 + \mathbf{V}_2 \times \mathbf{V}_3$$

$$\mathbf{V}_1 \times (\mathbf{V}_2 \times \mathbf{V}_3) = \mathbf{V}_2 (\mathbf{V}_3 \cdot \mathbf{V}_1) - \mathbf{V}_3 (\mathbf{V}_1 \cdot \mathbf{V}_2)$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0} \text{ (zero vector)}$$

$$= \mathbf{0}$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

If $\mathbf{V}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$, $\mathbf{V}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$, $\mathbf{V}_3 = a_3\mathbf{i} + b_3\mathbf{j} + c_3\mathbf{k}$,
then

$$\mathbf{V}_1 \times \mathbf{V}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (b_1c_2 - b_2c_1)\mathbf{i} + (c_1a_2 - c_2a_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

It should be noted that, since $\mathbf{V}_1 \times \mathbf{V}_2 = -\mathbf{V}_2 \times \mathbf{V}_1$, the vector product is not commutative.

Scalar Triple Product

There is only one possible interpretation of the expression $\mathbf{V}_1 \cdot \mathbf{V}_2 \times \mathbf{V}_3$ and that is $\mathbf{V}_1 \cdot (\mathbf{V}_2 \times \mathbf{V}_3)$ which is obviously a scalar.

Further $\mathbf{V}_1 \cdot (\mathbf{V}_2 \times \mathbf{V}_3) = (\mathbf{V}_1 \times \mathbf{V}_2) \cdot \mathbf{V}_3 = \mathbf{V}_2 \cdot (\mathbf{V}_3 \times \mathbf{V}_1)$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ = r_1 r_2 r_3 \cos \phi \sin \theta,$$

Where θ is the angle between $\mathbf{V}_2$ and $\mathbf{V}_3$ and ϕ is the angle between $\mathbf{V}_1$ and the normal to the plane of $\mathbf{V}_2$ and $\mathbf{V}_3$. This product is called the *scalar triple product* and is written as $[\mathbf{V}_1 \mathbf{V}_2 \mathbf{V}_3]$.

The determinant indicates that it can be considered as the volume of the parallelepiped whose three determining edges are $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$.

It also follows that cyclic permutation of the subscripts does not change the value of the scalar triple product so that

$$[\mathbf{V}_1 \mathbf{V}_2 \mathbf{V}_3] = [\mathbf{V}_2 \mathbf{V}_3 \mathbf{V}_1] = [\mathbf{V}_3 \mathbf{V}_1 \mathbf{V}_2]$$

$$\text{but } [\mathbf{V}_1 \mathbf{V}_2 \mathbf{V}_3] = -[\mathbf{V}_2 \mathbf{V}_1 \mathbf{V}_3] \text{ etc. and } [\mathbf{V}_1 \mathbf{V}_1 \mathbf{V}_2] \equiv 0 \text{ etc.}$$

Given three non-coplanar reference vectors $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$, the *reciprocal system* is given by $\mathbf{V}_1^*$, $\mathbf{V}_2^*$ and $\mathbf{V}_3^*$, where

$$1 = v_1 v_1^* = v_2 v_2^* = v_3 v_3^*$$

$$0 = v_1 v_2^* = v_1 v_3^* = v_2 v_1^* \text{ etc.}$$

$$\mathbf{V}_1^* = \frac{\mathbf{V}_2 \times \mathbf{V}_3}{[\mathbf{V}_1 \mathbf{V}_2 \mathbf{V}_3]}, \quad \mathbf{V}_2^* = \frac{\mathbf{V}_3 \times \mathbf{V}_1}{[\mathbf{V}_1 \mathbf{V}_2 \mathbf{V}_3]}, \quad \mathbf{V}_3^* = \frac{\mathbf{V}_1 \times \mathbf{V}_2}{[\mathbf{V}_1 \mathbf{V}_2 \mathbf{V}_3]}$$

The system $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is its own reciprocal.

Vector Triple Product

The product $\mathbf{V}_1 \times (\mathbf{V}_2 \times \mathbf{V}_3)$ defines the *vector triple product*. Obviously, in this case, the brackets are vital to the definition.

$$\mathbf{V}_1 \times (\mathbf{V}_2 \times \mathbf{V}_3) = (\mathbf{V}_1 \cdot \mathbf{V}_3)\mathbf{V}_2 - (\mathbf{V}_1 \cdot \mathbf{V}_2)\mathbf{V}_3 \\ = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} & \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} & \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \end{vmatrix}$$

i.e. it is a vector, perpendicular to $\mathbf{V}_1$, lying in the plane of $\mathbf{V}_2$, $\mathbf{V}_3$.

Similarly

$$(\mathbf{V}_1 \times \mathbf{V}_2) \times \mathbf{V}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} & \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} & \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \\ a_3 & b_3 & c_3 \end{vmatrix} \\ \mathbf{V}_1 \times (\mathbf{V}_2 \times \mathbf{V}_3) + \mathbf{V}_2 \times (\mathbf{V}_3 \times \mathbf{V}_1) + \mathbf{V}_3 \times (\mathbf{V}_1 \times \mathbf{V}_2) \equiv 0$$

If $\mathbf{V}_1 \times (\mathbf{V}_2 \times \mathbf{V}_3) = (\mathbf{V}_1 \times \mathbf{V}_2) \times \mathbf{V}_3$ then $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$ form an *orthogonal set*. Thus $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ form an orthogonal set.

Geometry of the Plane, Straight Line and Sphere

The position vectors of the fixed points A , B , C , D relative to O are $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, $\mathbf{V}_4$ and the position vector of the variable point P is $\mathbf{V}$.

The vector form of the equation of the straight line through A parallel to $\mathbf{V}_2$ is

$$\mathbf{V} = \mathbf{V}_1 + r\mathbf{V}_2$$

$$\text{or } (\mathbf{V} - \mathbf{V}_1) = r\mathbf{V}_2$$

$$\text{or } (\mathbf{V} - \mathbf{V}_1) \times \mathbf{V}_2 = 0$$

while that of the plane through A perpendicular to $\mathbf{V}_2$ is

$$(\mathbf{V} - \mathbf{V}_1) \cdot \mathbf{V}_2 = 0$$

The equation of the line AB is

$$\mathbf{V} = r\mathbf{V}_1 + (1 - r)\mathbf{V}_2$$

and those of the bisectors of the angles between $\mathbf{V}_1$ and $\mathbf{V}_2$ are

$$\mathbf{V} = r \left(\frac{\mathbf{V}_1}{v_1} \pm \frac{\mathbf{V}_2}{v_2} \right)$$

$$\text{or } \mathbf{V} = r(\hat{\mathbf{v}}_1 \pm \hat{\mathbf{v}}_2)$$

The perpendicular from C to the line through A parallel to $\mathbf{V}_2$ has as its equation

$$\mathbf{V} = \mathbf{V}_1 - \mathbf{V}_3 - \hat{\mathbf{v}}_2 \cdot (\mathbf{V}_1 - \mathbf{V}_3) \hat{\mathbf{v}}_2.$$

The condition for the intersection of the two lines,

$$\mathbf{V} = \mathbf{V}_1 + r\mathbf{V}_3$$

$$\text{and } \mathbf{V} = \mathbf{V}_2 + s\mathbf{V}_4$$

$$\text{is } [(\mathbf{V}_1 - \mathbf{V}_2)\mathbf{V}_3\mathbf{V}_4] = 0.$$

The common perpendicular to the above two lines is the line of intersection of the two planes

$$[(\mathbf{V} - \mathbf{V}_1)\mathbf{V}_3(\mathbf{V}_3 \times \mathbf{V}_4)] = 0$$

$$\text{and } [(\mathbf{V} - \mathbf{V}_2)\mathbf{V}_4(\mathbf{V}_3 \times \mathbf{V}_4)] = 0$$

and the length of this perpendicular is

$$\frac{[(\mathbf{V}_1 - \mathbf{V}_2)\mathbf{V}_3\mathbf{V}_4]}{|\mathbf{V}_3 \times \mathbf{V}_4|}.$$

The equation of the line perpendicular to the plane ABC is

$$\mathbf{V} = \mathbf{V}_1 \times \mathbf{V}_2 + \mathbf{V}_2 \times \mathbf{V}_3 + \mathbf{V}_3 \times \mathbf{V}_1$$

and the distance of the plane from the origin is

$$\frac{[\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3]}{|(\mathbf{V}_2 - \mathbf{V}_1) \times (\mathbf{V}_3 - \mathbf{V}_1)|}.$$

In general the vector equation

$$\mathbf{V} \cdot \mathbf{V}_2 = r$$

defines the plane which is perpendicular to $\mathbf{V}_2$, and the perpendicular distance from A to this plane is

$$\frac{r - \mathbf{V}_1 \cdot \mathbf{V}_2}{v_2}$$

The distance from A , measured along a line parallel to $\mathbf{V}_3$, is

$$\frac{r - \mathbf{V}_1 \cdot \mathbf{V}_2}{\mathbf{V}_2 \cdot \hat{\mathbf{v}}_3} \quad \text{or} \quad \frac{r - \mathbf{V}_1 \cdot \mathbf{V}_2}{v_2 \cos \theta}$$

where θ is the angle between $\mathbf{V}_2$ and $\mathbf{V}_3$.

(If this plane contains the point C then $r = \mathbf{V}_3 \cdot \mathbf{V}_2$ and if it passes through the origin then $r = 0$.)

Given two planes

$$\mathbf{V} \cdot \mathbf{V}_1 = r$$

$$\mathbf{V} \cdot \mathbf{V}_2 = s$$

then any plane through the line of intersection of these two planes is given by

$$\mathbf{V} \cdot (\mathbf{V}_1 + \lambda \mathbf{V}_2) = r + \lambda s$$

where λ is a scalar parameter. In particular $\lambda = \pm v_1/v_2$ yields the equation of the two planes bisecting the angle between the given planes.

The plane through A parallel to the plane of $\mathbf{V}_2, \mathbf{V}_3$ is

$$\mathbf{V} = \mathbf{V}_1 + r\mathbf{V}_2 + s\mathbf{V}_3$$

$$\text{or } (\mathbf{V} - \mathbf{V}_1) \cdot \mathbf{V}_2 \times \mathbf{V}_3 = 0$$

$$\text{or } [\mathbf{V}\mathbf{V}_2\mathbf{V}_3] - [\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3] = 0$$

so that the expansion in rectangular Cartesian coordinates yields

$$\begin{vmatrix} (x - a_1) & (y - b_1) & (z - c_1) \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \quad (\mathbf{V} \equiv x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

which is obviously the usual linear equation in x, y and z .

The plane through AB parallel to $\mathbf{V}_3$ is given by

$$[(\mathbf{V} - \mathbf{V}_1)(\mathbf{V}_1 - \mathbf{V}_2)\mathbf{V}_3] = 0$$

$$\text{or } [\mathbf{V}\mathbf{V}_2\mathbf{V}_3] - [\mathbf{V}\mathbf{V}_1\mathbf{V}_3] - [\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3] = 0$$

The plane through the three points A, B and C is

$$\mathbf{V} = \mathbf{V}_1 + s(\mathbf{V}_2 - \mathbf{V}_1) + t(\mathbf{V}_3 - \mathbf{V}_1)$$

$$\begin{aligned}
&\text{or} \quad \mathbf{V} = r\mathbf{V}_1 + s\mathbf{V}_2 + t\mathbf{V}_3 \quad (r + s + t \equiv 1) \\
&\text{or} \quad [(\mathbf{V} - \mathbf{V}_1)(\mathbf{V}_1 - \mathbf{V}_2)(\mathbf{V}_2 - \mathbf{V}_3)] = 0 \\
&\text{or} \quad [\mathbf{V}\mathbf{V}_1\mathbf{V}_2] + [\mathbf{V}\mathbf{V}_2\mathbf{V}_3] + [\mathbf{V}\mathbf{V}_3\mathbf{V}_1] - [\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3] = 0
\end{aligned}$$

For four points A, B, C, D to be coplanar, then

$$r\mathbf{V}_1 + s\mathbf{V}_2 + t\mathbf{V}_3 + u\mathbf{V}_4 \equiv 0 \equiv r + s + t + u$$

The following formulae relate to a sphere when the vectors are taken to lie in three dimensional space and to a circle when the space is two dimensional. For a circle in three dimensions take the intersection of the sphere with a plane.

The equation of a sphere with center O and radius OA is

$$\begin{aligned}
&\mathbf{V} \cdot \mathbf{V} = v_1^2 \quad (\text{not } \mathbf{V} = \mathbf{V}_1) \\
&\text{or} \quad (\mathbf{V} - \mathbf{V}_1) \cdot (\mathbf{V} + \mathbf{V}_1) = 0
\end{aligned}$$

while that of a sphere with center B radius v_1 is

$$\begin{aligned}
&(\mathbf{V} - \mathbf{V}_2) \cdot (\mathbf{V} - \mathbf{V}_2) = v_1^2 \\
&\text{or} \quad \mathbf{V} \cdot (\mathbf{V} - 2\mathbf{V}_2) = v_1^2 - v_2^2
\end{aligned}$$

If the above sphere passes through the origin then

$$\mathbf{V} \cdot (\mathbf{V} - 2\mathbf{V}_2) = 0$$

(note that in two dimensional polar coordinates this is simply)

$$r = 2a \cdot \cos \theta$$

while in three dimensional Cartesian coordinates it is

$$x^2 + y^2 + z^2 - 2(a_2x + b_2y + c_2z) = 0.$$

The equation of a sphere having the points A and B as the extremities of a diameter is

$$(\mathbf{V} - \mathbf{V}_1) \cdot (\mathbf{V} - \mathbf{V}_2) = 0.$$

The square of the length of the tangent from C to the sphere with center B and radius v_1 is given by

$$(\mathbf{V}_3 - \mathbf{V}_2) \cdot (\mathbf{V}_3 - \mathbf{V}_2) = v_1^2$$

The condition that the plane $\mathbf{V} \cdot \mathbf{V}_3 = s$ is tangential to the sphere $(\mathbf{V} - \mathbf{V}_2) \cdot (\mathbf{V} - \mathbf{V}_2) = v_1^2$ is

$$(s - \mathbf{V}_3 \cdot \mathbf{V}_2) \cdot (s - \mathbf{V}_3 \cdot \mathbf{V}_2) = v_1^2 v_3^2.$$

The equation of the tangent plane at D , on the surface of sphere $(\mathbf{V} - \mathbf{V}_2) \cdot (\mathbf{V} - \mathbf{V}_2) = v_1^2$, is

$$\begin{aligned}
&(\mathbf{V} - \mathbf{V}_4) \cdot (\mathbf{V}_4 - \mathbf{V}_2) = 0 \\
&\text{or} \quad \mathbf{V} \cdot \mathbf{V}_4 - \mathbf{V}_2 \cdot (\mathbf{V} + \mathbf{V}_4) = v_1^2 - v_2^2
\end{aligned}$$

The condition that the two circles $(\mathbf{V} - \mathbf{V}_2) \cdot (\mathbf{V} - \mathbf{V}_2) = v_1^2$ and $(\mathbf{V} - \mathbf{V}_4) \cdot (\mathbf{V} - \mathbf{V}_4) = v_3^2$ intersect orthogonally is clearly

$$(\mathbf{V}_2 - \mathbf{V}_4) \cdot (\mathbf{V}_2 - \mathbf{V}_4) = v_1^2 + v_3^2$$

The polar plane of D with respect to the circle

$$\begin{aligned}
&(\mathbf{V} - \mathbf{V}_2) \cdot (\mathbf{V} - \mathbf{V}_2) = v_1^2 \quad \text{is} \\
&\mathbf{V} \cdot \mathbf{V}_4 - \mathbf{V}_2 \cdot (\mathbf{V} + \mathbf{V}_4) = v_1^2 - v_2^2
\end{aligned}$$

Any sphere through the intersection of the two spheres $(\mathbf{V} - \mathbf{V}_2) \cdot (\mathbf{V} - \mathbf{V}_2) = v_1^2$ and $(\mathbf{V} - \mathbf{V}_4) \cdot (\mathbf{V} - \mathbf{V}_4) = v_3^2$ is given by

$$(\mathbf{V} - \mathbf{V}_2) \cdot (\mathbf{V} - \mathbf{V}_2) + \lambda(\mathbf{V} - \mathbf{V}_4) \cdot (\mathbf{V} - \mathbf{V}_4) = v_1^2 + \lambda v_3^2$$

while the radical plane of two such spheres is

$$\mathbf{V} \cdot (\mathbf{V}_2 - \mathbf{V}_4) = -\frac{1}{2}(v_1^2 - v_2^2 - v_3^2 + v_4^2)$$

Differentiation of Vectors

If $\mathbf{V}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$, and $\mathbf{V}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$, and if $\mathbf{V}_1$ and $\mathbf{V}_2$ are functions of the scalar t , then

$$\begin{aligned}
\frac{d}{dt}(\mathbf{V}_1 + \mathbf{V}_2 + \dots) &= \frac{d\mathbf{V}_1}{dt} + \frac{d\mathbf{V}_2}{dt} + \dots, \\
\text{where } \frac{d\mathbf{V}_1}{dt} &= \frac{da_1}{dt}\mathbf{i} + \frac{db_1}{dt}\mathbf{j} + \frac{dc_1}{dt}\mathbf{k}, \text{ etc.} \\
\frac{d}{dt}(\mathbf{V}_1 \cdot \mathbf{V}_2) &= \frac{d\mathbf{V}_1}{dt} \cdot \mathbf{V}_2 + \mathbf{V}_1 \cdot \frac{d\mathbf{V}_2}{dt} \\
\frac{d}{dt}(\mathbf{V}_1 \times \mathbf{V}_2) &= \frac{d\mathbf{V}_1}{dt} \times \mathbf{V}_2 + \mathbf{V}_1 \times \frac{d\mathbf{V}_2}{dt} \\
\mathbf{V} \cdot \frac{d\mathbf{V}}{dt} &= v \cdot \frac{dv}{dt}
\end{aligned}$$

In particular, if $\mathbf{V}$ is a vector of constant length then the right hand side of the last equation is identically zero showing that $\mathbf{V}$ is perpendicular to its derivative.

The derivatives of the triple products are

$$\frac{d}{dt}[\mathbf{V}_1 \mathbf{V}_2 \mathbf{V}_3] = \left[\left(\frac{d\mathbf{V}_1}{dt} \right) \mathbf{V}_2 \mathbf{V}_3 \right] + \left[\mathbf{V}_1 \left(\frac{d\mathbf{V}_2}{dt} \right) \mathbf{V}_3 \right] + \left[\mathbf{V}_1 \mathbf{V}_2 \left(\frac{d\mathbf{V}_3}{dt} \right) \right]$$

and $\frac{d}{dt}(\mathbf{V}_1 \times (\mathbf{V}_2 \times \mathbf{V}_3)) = \left(\frac{d\mathbf{V}_1}{dt} \right) \times (\mathbf{V}_2 \times \mathbf{V}_3) + \mathbf{V}_1 \times \left(\left(\frac{d\mathbf{V}_2}{dt} \right) \times \mathbf{V}_3 \right) + \mathbf{V}_1 \times \left(\mathbf{V}_2 \times \left(\frac{d\mathbf{V}_3}{dt} \right) \right)$

Geometry of Curves in Space

s = the length of arc, measured from some fixed point on the curve (fig. 3).

$\mathbf{V}_1$ = the position vector of the point A on the curve

$\mathbf{V}_1 + \delta\mathbf{V}_1$ = the position vector of the point P in the neighborhood of A

$\hat{\mathbf{t}}$ = the unit tangent to the curve at the point A , measured in the direction of s increasing.

The normal plane is that plane which is perpendicular to the unit tangent. The principal normal is defined as the intersection of the normal plane with the plane defined by $\mathbf{V}_1$ and $\mathbf{V}_1 + \delta\mathbf{V}_1$ in the limit as $\delta\mathbf{V}_1 \rightarrow 0$.

$\hat{\mathbf{n}}$ = the unit normal (principal) at the point A . The plane defined by $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ is called the *osculating plane* (alternatively plane of curvature or local plane).

ρ = the radius of curvature at A .

$\delta\theta$ = the angle subtended at the origin by $\delta\mathbf{V}_1$.

$$\kappa = \frac{d\theta}{ds} = \frac{1}{\rho}$$

$\hat{\mathbf{b}}$ = the unit binormal i.e. the unit vector which is parallel to $\hat{\mathbf{t}} \times \hat{\mathbf{n}}$ at the point A :

λ = the torsion of the curve at A

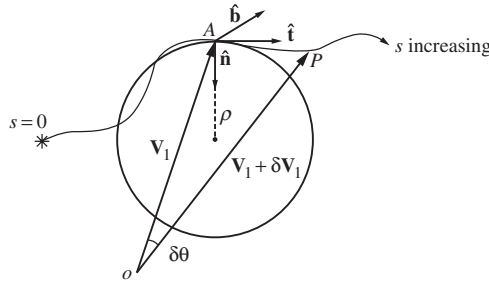


Figure 3.

Frenet's Formulae:

$$\begin{aligned} \frac{d\hat{\mathbf{t}}}{ds} &= \kappa \hat{\mathbf{n}} \\ \frac{d\hat{\mathbf{n}}}{ds} &= -\kappa \hat{\mathbf{t}} + \lambda \hat{\mathbf{b}} \\ \frac{d\hat{\mathbf{b}}}{ds} &= -\lambda \hat{\mathbf{n}} \end{aligned}$$

The following formulae are also applicable:

Unit tangent	$\hat{\mathbf{t}} = \frac{d\mathbf{V}_1}{ds}$
Equation of the tangent	$(\mathbf{V} - \mathbf{V}_1) \times \hat{\mathbf{t}} = 0$
or	$\mathbf{V} = \mathbf{V}_1 + q\hat{\mathbf{t}}$
Unit normal	$\hat{\mathbf{n}} = \frac{1}{\kappa} \frac{d^2\mathbf{V}_1}{ds^2}$
Equation of the normal plane	$(\mathbf{V} - \mathbf{V}_1) \cdot \hat{\mathbf{t}} = 0$
Equation of the normal	$(\mathbf{V} - \mathbf{V}_1) \times \hat{\mathbf{n}} = 0$
or	$\mathbf{V} = \mathbf{V}_1 + r\hat{\mathbf{n}}$
Unit binormal	$\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$
Equation of the binormal	$(\mathbf{V} - \mathbf{V}_1) \times \hat{\mathbf{b}} = 0$
or	$\mathbf{V} = \mathbf{V}_1 + u\hat{\mathbf{b}}$
or	$\mathbf{V} = \mathbf{V}_1 + w \frac{d\mathbf{V}_1}{ds} \times \frac{d^2\mathbf{V}_1}{ds^2}$
Equation of the osculating plane:	$[(\mathbf{V} - \mathbf{V}_1)\hat{\mathbf{t}}\hat{\mathbf{n}}] = 0$
or	$\left[(\mathbf{V} - \mathbf{V}_1) \left(\frac{d\mathbf{V}_1}{ds} \right) \left(\frac{d^2\mathbf{V}_1}{ds^2} \right) \right] = 0$

A *geodetic line* on a surface is a curve, the osculating plane of which is everywhere normal to the surface.

The differential equation of the geodetic is

$$[\hat{\mathbf{n}} d\mathbf{V}_1 d^2\mathbf{V}_1] = 0$$

Differential Operators—Rectangular Coordinates

$$dS = \frac{\partial S}{\partial x} \cdot dx + \frac{\partial S}{\partial y} \cdot dy + \frac{\partial S}{\partial z} \cdot dz$$

By definition

$$\nabla \equiv \text{del} \equiv \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

$$\nabla^2 \equiv \text{Laplacian} \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

If S is a scalar function, then

$$\nabla S \equiv \text{grad } S \equiv \frac{\partial S}{\partial x} \mathbf{i} + \frac{\partial S}{\partial y} \mathbf{j} + \frac{\partial S}{\partial z} \mathbf{k}$$

Grad S defines both the direction and magnitude of the maximum rate of increase of S at any point. Hence the name *gradient* and also its vectorial nature. ∇S is independent of the choice of rectangular coordinates.

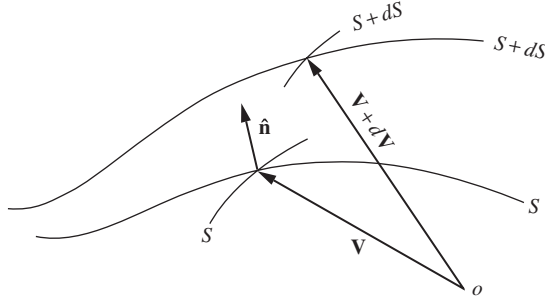


Figure 4.

$$\nabla S = \frac{\partial S}{\partial n} \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is the unit normal to the surface $S = \text{constant}$, in the direction of S increasing. The total derivative of S at a point having the position vector $\mathbf{V}$ is given by (fig. 4)

$$\begin{aligned} dS &= \frac{\partial S}{\partial n} \hat{\mathbf{n}} \cdot d\mathbf{V} \\ &= d\mathbf{V} \cdot \nabla S \end{aligned}$$

and the directional derivative of S in the direction of $\mathbf{U}$ is

$$\mathbf{U} \cdot \nabla S = \mathbf{U} \cdot (\nabla S) = (\mathbf{U} \cdot \nabla) S$$

Similarly the directional derivative of the vector $\mathbf{V}$ in the direction of $\mathbf{U}$ is

$$(\mathbf{U} \cdot \nabla) \mathbf{V}$$

The *distributive* law holds for finding a gradient. Thus if S and T are scalar functions

$$\nabla(S + T) = \nabla S + \nabla T$$

The *associative* law becomes the rule for differentiating a product:

$$\nabla(ST) = S\nabla T + T\nabla S$$

If $\mathbf{V}$ is a vector function with the magnitudes of the components parallel to the three coordinate axes V_x, V_y, V_z , then

$$\nabla \cdot \mathbf{V} \equiv \text{div } \mathbf{V} \equiv \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

The divergence obeys the distributive law. Thus, if $\mathbf{V}$ and $\mathbf{U}$ are vector functions, then

$$\nabla \cdot (\mathbf{V} + \mathbf{U}) = \nabla \cdot \mathbf{V} + \nabla \cdot \mathbf{U}$$

$$\nabla \cdot (S\mathbf{V}) = (\nabla S) \cdot \mathbf{V} + S(\nabla \cdot \mathbf{V})$$

$$\nabla \cdot (\mathbf{U} \times \mathbf{V}) = \mathbf{V} \cdot (\nabla \times \mathbf{U}) - \mathbf{U} \cdot (\nabla \times \mathbf{V})$$

As with the gradient of a scalar, the divergence of a vector is invariant under a transformation from one set of rectangular coordinates to another.

$$\begin{aligned} \nabla \times \mathbf{V} &\equiv \text{curl } \mathbf{V} \text{ (sometimes } \nabla \wedge \mathbf{V} \text{ or } \text{rot } \mathbf{V}) \\ &\equiv \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} \end{aligned}$$

The *curl* (or *rotation*) of a vector is a vector which is invariant under a transformation from one set of rectangular coordinates to another.

$$\begin{aligned}\nabla \times (\mathbf{U} + \mathbf{V}) &= \nabla \times \mathbf{U} + \nabla \times \mathbf{V} \\ \nabla \times (S\mathbf{V}) &= (\nabla S) \times \mathbf{V} + S(\nabla \times \mathbf{V}) \\ \nabla \times (\mathbf{U} \times \mathbf{V}) &= (\mathbf{V} \cdot \nabla)\mathbf{U} - (\mathbf{U} \cdot \nabla)\mathbf{V} + \mathbf{U}(\nabla \cdot \mathbf{V}) - \mathbf{V}(\nabla \cdot \mathbf{U}) \\ \text{grad}(\mathbf{U} \cdot \mathbf{V}) &= \nabla(\mathbf{U} \cdot \mathbf{V}) \\ &= (\mathbf{V} \cdot \nabla)\mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{V} + \mathbf{V} \times (\nabla \times \mathbf{U}) + \mathbf{U} \times (\nabla \times \mathbf{V})\end{aligned}$$

If

$$\begin{aligned}\mathbf{V} &= V_x \mathbf{i} + V_y \mathbf{j} + V_z \mathbf{k} \\ \nabla \cdot \mathbf{V} &= \nabla V_x \cdot \mathbf{i} + \nabla V_y \cdot \mathbf{j} + \nabla V_z \cdot \mathbf{k} \\ \text{and } \nabla \times \mathbf{V} &= \nabla V_x \times \mathbf{i} + \nabla V_y \times \mathbf{j} + \nabla V_z \times \mathbf{k}\end{aligned}$$

The operator ∇ can be used more than once. The number of possibilities where ∇ is used twice are

$$\begin{aligned}\nabla \cdot (\nabla \theta) &\equiv \text{div grad } \theta \\ \nabla \times (\nabla \theta) &\equiv \text{curl grad } \theta \\ \nabla(\nabla \cdot \mathbf{V}) &\equiv \text{grad div } \mathbf{V} \\ \nabla \cdot (\nabla \times \mathbf{V}) &\equiv \text{div curl } \mathbf{V} \\ \nabla \times (\nabla \times \mathbf{V}) &\equiv \text{curl curl } \mathbf{V}\end{aligned}$$

Thus:

$$\begin{aligned}\text{div grad } S &\equiv \nabla \cdot (\nabla S) \equiv \text{Laplacian } S \equiv \nabla^2 S \\ &\equiv \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2}\end{aligned}$$

$\text{curl grad } S \equiv 0$;

$$\begin{aligned}\text{curl curl } \mathbf{V} &\equiv \text{grad div } \mathbf{V} - \nabla^2 \mathbf{V}; \\ \text{div curl } \mathbf{V} &\equiv 0\end{aligned}$$

Taylor's expansion in three dimensions can be written

$$\begin{aligned}\text{where } f(\mathbf{V} + \varepsilon) &= e^{\varepsilon \cdot \nabla} f(\mathbf{V}) \\ \text{and } \mathbf{V} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ \varepsilon &= h\mathbf{i} + l\mathbf{j} + m\mathbf{k}\end{aligned}$$

(note the analogy with $f_p = e^{phD} f_0$ in finite difference methods).

Orthogonal Curvilinear Coordinates

If at a point P there exist three uniform point functions u , v and w so that the surfaces $u = \text{const.}$, $v = \text{const.}$, and $w = \text{const.}$, intersect in three distinct curves through P then the surfaces are called the *coordinate surfaces* through P . The three lines of intersection are referred to as the *coordinate lines* and their tangents a , b , and c as the *coordinate axes*. When the coordinate axes form an orthogonal set the system is said to define *orthogonal curvilinear coordinates* at P .

Consider an infinitesimal volume enclosed by the surfaces u , v , w , $u + du$, $v + dv$, and $w + dw$ (fig. 5).

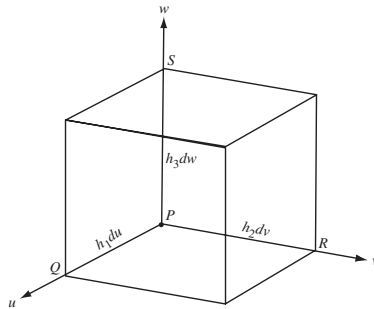


Figure 5.

The surface $PRQS \equiv u = \text{const.}$, and the face of the curvilinear figure immediately opposite this is $u + du = \text{const.}$ etc.

In terms of these surface constants

$$\begin{aligned}P &= P(u, v, w) \\ Q &= Q(u + du, v, w) \quad \text{and} \quad PQ = h_1 du\end{aligned}$$

$$\begin{aligned} R &= R(u, v + dv, w) & PR &= h_2 dv \\ S &= S(u, v, w + dw) & PS &= h_3 dw \end{aligned}$$

where h_1 , h_2 , and h_3 are functions of u , v , and w .

In rectangular Cartesians **i, j, k**

$$h_1 = 1, \quad h_2 = 1, \quad h_3 = 1.$$

$$\frac{\hat{\mathbf{a}}}{h_1} \frac{\partial}{\partial u} = \mathbf{i} \frac{\partial}{\partial x}, \quad \frac{\hat{\mathbf{b}}}{h_2} \frac{\partial}{\partial v} = \mathbf{j} \frac{\partial}{\partial y}, \quad \frac{\hat{\mathbf{c}}}{h_3} \frac{\partial}{\partial w} = \mathbf{k} \frac{\partial}{\partial z}$$

In cylindrical coordinates **$\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{k}}$**

$$h_1 = 1, \quad h_2 = r, \quad h_3 = 1.$$

$$\frac{\hat{\mathbf{a}}}{h_1} \frac{\partial}{\partial u} = \hat{\mathbf{r}} \frac{\partial}{\partial r}, \quad \frac{\hat{\mathbf{b}}}{h_2} \frac{\partial}{\partial v} = \frac{\hat{\boldsymbol{\phi}}}{r} \frac{\partial}{\partial \phi}, \quad \frac{\hat{\mathbf{c}}}{h_3} \frac{\partial}{\partial w} = \hat{\mathbf{k}} \frac{\partial}{\partial z}$$

In spherical coordinates **$\hat{\mathbf{r}}, \hat{u}, \hat{\boldsymbol{\phi}}$**

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r \sin \theta$$

$$\frac{\hat{\mathbf{a}}}{h_1} \frac{\partial}{\partial u} = \hat{\mathbf{r}} \frac{\partial}{\partial r}, \quad \frac{\hat{\mathbf{b}}}{h_2} \frac{\partial}{\partial v} = \frac{\hat{\boldsymbol{\phi}}}{r} \frac{\partial}{\partial \theta}, \quad \frac{\hat{\mathbf{c}}}{h_3} \frac{\partial}{\partial w} = \frac{\hat{\boldsymbol{\phi}}}{r \sin \theta} \frac{\partial}{\partial \phi}$$

The general expressions for grad, div and curl together with those for ∇^2 and the directional derivative are, in orthogonal curvilinear coordinates, given by

$$\begin{aligned} \nabla S &= \frac{\hat{\mathbf{a}}}{h_1} \frac{\partial S}{\partial u} + \frac{\hat{\mathbf{b}}}{h_2} \frac{\partial S}{\partial v} + \frac{\hat{\mathbf{c}}}{h_3} \frac{\partial S}{\partial w} \\ (\mathbf{V} \cdot \nabla) S &= \frac{V_1}{h_1} \frac{\partial S}{\partial u} + \frac{V_2}{h_2} \frac{\partial S}{\partial v} + \frac{V_3}{h_3} \frac{\partial S}{\partial w} \\ \nabla \cdot \mathbf{V} &= \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u} (h_2 h_3 V_1) + \frac{\partial}{\partial v} (h_3 h_1 V_2) + \frac{\partial}{\partial w} (h_1 h_2 V_3) \right\} \\ \nabla \times \mathbf{V} &= \frac{\hat{\mathbf{a}}}{h_2 h_3} \left\{ \frac{\partial}{\partial v} (h_3 V_3) - \frac{\partial}{\partial w} (h_2 V_2) \right\} + \frac{\hat{\mathbf{b}}}{h_3 h_1} \left\{ \frac{\partial}{\partial w} (h_1 V_1) - \frac{\partial}{\partial u} (h_3 V_3) \right\} \\ &\quad + \frac{\hat{\mathbf{c}}}{h_1 h_2} \left\{ \frac{\partial}{\partial u} (h_2 V_2) - \frac{\partial}{\partial v} (h_1 V_1) \right\} \\ \nabla^2 S &= \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u} \left(\frac{h_2 h_3}{h_1} \frac{\partial S}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{h_3 h_1}{h_2} \frac{\partial S}{\partial v} \right) + \frac{\partial}{\partial w} \left(\frac{h_1 h_2}{h_3} \frac{\partial S}{\partial w} \right) \right\} \end{aligned}$$

FORMULAS OF VECTOR ANALYSIS

	Rectangular coordinates	Cylindrical coordinates	Spherical coordinates
Conversion to rectangular coordinates		$x = r \cos \varphi \quad y = r \sin \varphi \quad z = z$	$x = r \cos \varphi \sin \theta \quad y = r \sin \varphi \sin \theta$ $z = r \cos \theta$
Gradient	$\nabla \phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}$	$\nabla \phi = \frac{\partial \phi}{\partial r} \mathbf{r} + \frac{1}{r} \frac{\partial \phi}{\partial \varphi} \boldsymbol{\phi} + \frac{\partial \phi}{\partial z} \mathbf{k}$	$\nabla \phi = \frac{\partial \phi}{\partial r} \mathbf{r} + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \boldsymbol{\theta} + \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \boldsymbol{\phi}$
Divergence	$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial (r A_r)}{\partial r} + \frac{1}{r} \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$	$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (A_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\varphi}{\partial \varphi}$
Curl	$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{1}{r} \mathbf{r} & \boldsymbol{\phi} & \frac{1}{r} \mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_r & r A_\varphi & A_z \end{vmatrix}$	$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{r} & \boldsymbol{\theta} & \boldsymbol{\phi} \\ \frac{1}{r^2} \sin \theta & \frac{r \sin \theta}{r} & \frac{\phi}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & r A_\theta & r A_\varphi \sin \theta \end{vmatrix}$
Laplacian	$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$	$\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2}$	$\nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2}$

Transformation of Integrals

s = the distance along some curve “ C ” in space and is measured from some fixed point.

S = a surface area

V = a volume contained by a specified surface

$\hat{\mathbf{t}}$ = the unit tangent to C at the point P

$\hat{\mathbf{n}}$ = the unit outward pointing normal

F = some vector function

ds = the vector element of curve ($= \hat{\mathbf{t}} ds$)

dS = the vector element of surface ($= \hat{\mathbf{n}} dS$)

Then

$$\int_{(c)} \mathbf{F} \cdot \hat{\mathbf{t}} ds = \int_{(c)} \mathbf{F} \cdot d\mathbf{s}$$

and when

$$\mathbf{F} = \nabla \phi$$

$$\int_{(c)} (\nabla \phi) \cdot \hat{\mathbf{t}} ds = \int_{(c)} d\phi$$

Gauss' Theorem (Green's Theorem)

When S defines a closed region having a volume V

$$\iiint_{(v)} (\nabla \cdot \mathbf{F}) dV = \iint_{(s)} (\mathbf{F} \cdot \hat{\mathbf{n}}) dS = \int \int_{(s)} \mathbf{F} \cdot d\mathbf{S}$$

also

$$\iiint_{(v)} (\nabla \phi) dV = \iint_{(s)} \phi \hat{\mathbf{n}} dS$$

and

$$\iiint_{(v)} (\nabla \times \mathbf{F}) dV = \iint_{(s)} (\hat{\mathbf{n}} \times \mathbf{F}) dS$$

Stokes' Theorem

When C is closed and bounds the open surface S .

$$\iint_{(s)} \hat{\mathbf{n}} \cdot (\nabla \times \mathbf{F}) dS = \int_{(c)} \mathbf{F} \cdot d\mathbf{s}$$

also

$$\iint_{(s)} (\hat{\mathbf{n}} \times \nabla \phi) dS = \int_{(c)} \phi d\mathbf{s}$$

Green's Theorem

$$\begin{aligned} \iint_{(s)} (\nabla \phi \cdot \nabla \theta) dS &= \iint_{(s)} \phi \hat{\mathbf{n}} \cdot (\nabla \theta) dS = \iiint_{(v)} \phi (\nabla^2 \theta) dV \\ &= \iint_{(s)} \theta \cdot \hat{\mathbf{n}} (\nabla \phi) dS = \iiint_{(v)} \theta (\nabla^2 \phi) dV \end{aligned}$$

MOMENT OF INERTIA FOR VARIOUS BODIES OF MASS

The mass of the body is indicated by m

Body	Axis	Moment of inertia	Body	Axis	Moment of inertia
Uniform thin rod	Normal to the length, at one end	$m \frac{l^2}{3}$	Spherical shell, very thin, mean radius, r	Any diameter	$m \frac{2}{3} r^2$
Uniform thin rod	Normal to the length, at the center	$m \frac{l^2}{12}$	Right circular cylinder of radius r , length l	The longitudinal axis of the solid	$m \frac{r^2}{2}$
The rectangular sheet, sides a and b	Through the center parallel to b	$m \frac{a^2}{12}$	Right circular cylinder of radius r , length l	Transverse diameter	$m \left(\frac{r^2}{4} + \frac{l^2}{12} \right)$
Thin rectangular sheet, sides a and b	Through the center perpendicular to the sheet	$m \frac{a^2 + b^2}{12}$	Hollow circular cylinder, length l , radii r_1 and r_2	The longitudinal axis of the figure	$m \frac{(r_1^2 + r_2^2)}{2}$
Thin circular sheet of radius r	Normal to the plate through the center	$m \frac{r^2}{2}$	Thin cylindrical shell, length l , mean radius, r	The longitudinal axis of the figure	mr^2
Thin circular sheet of radius r	Along any diameter	$m \frac{r^2}{4}$	Hollow circular cylinder, length l , radii r_1 and r_2	Transverse diameter	$m \left[\frac{r_1^2 + r_2^2}{4} + \frac{l^2}{12} \right]$
Thin circular ring. Radii r_1 and r_2	Through center normal to plane of ring	$m \frac{r_1^2 + r_2^2}{2}$	Hollow circular cylinder, length l , very thin, mean radius	Transverse diameter	$m \left(\frac{r^2}{2} + \frac{l^2}{12} \right)$
Thin circular ring. Radii r_1 and r_2	Any diameter	$m \frac{r_1^2 + r_2^2}{4}$	Elliptic cylinder, length l , transverse semiaxes a and b	Longitudinal axis	$m \left(\frac{a^2 + b^2}{4} \right)$
Rectangular parallelepiped, edges a , b , and c	Through center perpendicular to face ab , (parallel to edge c)	$m \frac{a^2 + b^2}{12}$	Right cone, altitude h , radius of base r	Axis of the figure	$m \frac{3}{10} r^2$
Sphere, radius r	Any diameter	$m \frac{2}{5} r^2$	Spheroid of revolution, equatorial radius r	Polar axis	$m \frac{2r^2}{5}$
Spherical shell, external radius, r_1 , internal radius r_2	Any diameter	$m \frac{2(r_1^5 - r_2^5)}{5(r_1^3 - r_2^3)}$	Ellipsoid, axes $2a$, $2b$, $2c$	Axis $2a$	$m \frac{(b^2 + c^2)}{5}$

Bessel Functions*

1. Bessel's differential equation for a real variable x is

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0$$

* From Beyer, W. H., Ed., *CRC Handbook of Mathematical Sciences*, 5th ed., CRC Press, Boca Raton, 1978, 500–503. With permission.

2. When n is not an integer, two independent solutions of the equation are $J_n(x)$, $J_{-n}(x)$, where

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

3. If n is an integer $J_n(x) = (-1)^n J_{-n}(x)$, where

$$J_n(x) = \frac{x^n}{2^n n!} \left\{ 1 - \frac{x^2}{2^2 \cdot 1!(n+1)} + \frac{x^4}{2^4 \cdot 2!(n+1)(n+2)} + \frac{x^6}{2^6 \cdot 3!(n+1)(n+2)(n+3)} + \dots \right\}$$

4. For $n=0$ and $n=1$, this formula becomes

$$\begin{aligned} J_0(x) &= 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \frac{x^8}{2^8(4!)^2} - \dots \\ J_1(x) &= \frac{x}{2} - \frac{x^3}{2^3 \cdot 1!2!} + \frac{x^5}{2^5 \cdot 2!3!} - \frac{x^7}{2^7 \cdot 3!4!} + \frac{x^9}{2^9 \cdot 4!5!} - \dots \end{aligned}$$

5. When x is large and positive, the following asymptotic series may be used

$$\begin{aligned} J_0(x) &= \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \left\{ P_0(x) \cos\left(x - \frac{\pi}{4}\right) - Q_0(x) \sin\left(x - \frac{\pi}{4}\right) \right\} \\ J_1(x) &= \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \left\{ P_1(x) \cos\left(x - \frac{3\pi}{4}\right) - Q_1(x) \sin\left(x - \frac{3\pi}{4}\right) \right\} \end{aligned}$$

where

$$\begin{aligned} P_0(x) &\sim 1 - \frac{1^2 \cdot 3^2}{2!(8x)^2} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{4!(8x)^4} - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot 11^2}{6!(8x)^6} + \dots \\ Q_0(x) &\sim -\frac{1^2}{1!8x} + \frac{1^2 \cdot 3^2 \cdot 5^2}{3!(8x)^3} - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2}{5!(8x)^5} + \dots \\ P_1(x) &\sim 1 + \frac{1^2 \cdot 3 \cdot 5}{2!(8x)^2} - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 9}{4!(8x)^4} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot 11 \cdot 13}{6!(8x)^6} - \dots \\ Q_1(x) &\sim \frac{1 \cdot 3}{1!8x} - \frac{1^2 \cdot 3^2 \cdot 5 \cdot 7}{3!(8x)^3} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9 \cdot 11}{5!(8x)^5} - \dots \end{aligned}$$

[In $P_1(x)$ the signs alternate from $+$ to $-$ after the first term]

6. If $x > 25$, it is convenient to use the formulas

$$\begin{aligned} J_0(x) &= A_0(x) \sin x + B_0(x) \cos x \\ J_1(x) &= B_1(x) \sin x - A_1(x) \cos x \end{aligned}$$

where

$$\begin{aligned} A_0(x) &= \frac{P_0(x) - Q_0(x)}{(\pi x)^{\frac{1}{2}}} \quad \text{and} \quad A_1(x) = \frac{P_1(x) - Q_1(x)}{(\pi x)^{\frac{1}{2}}} \\ B_0(x) &= \frac{P_0(x) + Q_0(x)}{(\pi x)^{\frac{1}{2}}} \quad \text{and} \quad B_1(x) = \frac{P_1(x) + Q_1(x)}{(\pi x)^{\frac{1}{2}}} \end{aligned}$$

7. The zeros of $J_0(x)$ and $J_1(x)$

If j_{0s} and j_{1s} are the s th zeros of $J_0(x)$ and $J_1(x)$ respectively, and if $a = 4_s - 1$, $b = 4_s + 1$

$$\begin{aligned} j_{0,s} &\sim \frac{1}{4} \pi a \left\{ 1 + \frac{2}{\pi^2 a^2} - \frac{62}{3\pi^4 a^4} + \frac{15,116}{15\pi^6 a^6} - \frac{12,554,474}{105\pi^8 a^8} + \frac{8,368,654,292}{315\pi^{10} a^{10}} - \dots \right\} \\ j_{1,s} &\sim \frac{1}{4} \pi b \left\{ 1 - \frac{6}{\pi^2 b^2} + \frac{6}{\pi^4 b^4} - \frac{4716}{5\pi^6 b^6} + \frac{3,902,418}{35\pi^8 b^8} - \frac{895,167,324}{35\pi^{10} b^{10}} + \dots \right\} \\ J_1(j_{0,s}) &\sim \frac{(-1)^{s+1} 2^{\frac{3}{2}}}{\pi a^{\frac{1}{2}}} \left\{ 1 - \frac{56}{3\pi^4 a^4} + \frac{9664}{5\pi^6 a^6} - \frac{7,381,280}{21\pi^8 a^8} + \dots \right\} \\ J_0(j_{1,s}) &\sim \frac{(-1)^s 2^{\frac{3}{2}}}{\pi b^{\frac{1}{2}}} \left\{ 1 + \frac{24}{\pi^4 b^4} - \frac{19,584}{10\pi^6 b^6} + \frac{2,466,720}{7\pi^8 b^8} - \dots \right\} \end{aligned}$$

8. Table of zeros for $J_0(x)$ and $J_1(x)$

$J_1(\alpha_n) = 0$		$J_0(\beta_n) = 0$	
Roots α_n	$J_1(\alpha_n)$	Roots β_n	$J_0(\beta_n)$
2.4048	0.5191	0.0000	1.0000
5.5201	-0.3403	3.8317	-0.4028
8.6537	0.2715	7.0156	0.3001
11.7915	-0.2325	10.1735	-0.2497
14.9309	0.2065	13.3237	0.2184
18.0711	-0.1877	16.4706	-0.1965
21.2116	0.1733	19.6159	0.1801

9. Recurrence formulas

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \quad nJ_n(x) + xJ'_n(x) = xJ_{n-1}(x)$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J'_n(x) \quad nJ_n(x) - xJ'_n(x) = xJ_{n+1}(x)$$

10. If J_n is written for $J_n(x)$ and $J_n^{(k)}$ is written for $\frac{d^k}{dx^k} \{J_n(x)\}$, then the following derivative relationships are important

$$J_0^{(r)} = -J_1^{(r-1)}$$

$$J_0^{(2)} = -J_0 + \frac{1}{x} J_1 = \frac{1}{2} (J_2 - J_0)$$

$$J_0^{(3)} = \frac{1}{x} J_0 + \left(1 - \frac{2}{x^2}\right) J_1 = \frac{1}{4} (-J_3 + 3J_1)$$

$$J_0^{(4)} = \left(1 - \frac{3}{x^2}\right) J_0 - \left(\frac{2}{x} - \frac{6}{x^3}\right) J_1 = \frac{1}{8} (J_4 - 4J_2 + 3J_0), \text{ etc.}$$

11. Half order Bessel functions

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

$$J_{n+\frac{3}{2}}(x) = -x^{n+\frac{1}{2}} \frac{d}{dx} \{x^{-(n+\frac{1}{2})} J_{n+\frac{1}{2}}(x)\}$$

$$J_{n-\frac{1}{2}}(x) = x^{-(n+\frac{1}{2})} \frac{d}{dx} \{x^{n+\frac{1}{2}} J_{n+\frac{1}{2}}(x)\}$$

n	$\left(\frac{\pi x}{2}\right)^{\frac{1}{2}} J_{n+\frac{1}{2}}(x)$	$\left(\frac{\pi x}{2}\right)^{\frac{1}{2}} J_{-(n+\frac{1}{2})}(x)$
0	$\sin x$	$\cos x$
1	$\frac{\sin x}{x} - \cos x$	$-\frac{\cos x}{x} - \sin x$
2	$\left(\frac{3}{x^2} - 1\right) \sin x - \frac{3}{x} \cos x$	$\left(\frac{3}{x^2} - 1\right) \cos x + \frac{3}{x} \sin x$
3	$\left(\frac{15}{x^3} - \frac{6}{x}\right) \sin x - \left(\frac{15}{x^2} - 1\right) \cos x$	$-\left(\frac{15}{x^3} - \frac{6}{x}\right) \cos x - \left(\frac{15}{x^2} - 1\right) \sin x$
	etc.	

12. Additional solutions to Bessel's equation are

$$Y_n(x) \quad (\text{also called Weber's function, and sometimes denoted by } N_n(x))$$

$$H_n^{(1)}(x) \quad \text{and} \quad H_n^{(2)}(x) \quad (\text{also called Hankel functions})$$

These solutions are defined as follows

$$Y_n(x) = \begin{cases} \frac{J_n(x) \cos(n\pi) - J_{-n}(x)}{\sin(n\pi)} & n \text{ not an integer} & H_n^{(1)}(x) = J_n(x) + iY_n(x) \\ \lim_{v \rightarrow n} \frac{J_v(x) \cos(v\pi) - J_{-v}(x)}{\sin(v\pi)} & n \text{ an integer} & H_n^{(2)}(x) = J_n(x) - iY_n(x) \end{cases}$$

The additional properties of these functions may all be derived from the above relations and the known properties of $J_n(x)$.

13. Complete solutions to Bessel's equation may be written as

$$\text{or} \quad c_1 J_n(x) + c_2 J_{-n}(x) \quad \text{if } n \text{ is not an integer}$$

$$\text{or} \quad \left. \begin{aligned} c_1 J_n(x) + c_2 Y_n(x) \\ c_1 H_n^{(1)}(x) + c_2 H_n^{(2)}(x) \end{aligned} \right\} \text{for any value of } n$$

14. The modified (or hyperbolic) Bessel's differential equation is

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + n^2)y = 0$$

15. When n is not an integer, two independent solutions of the equation are $I_n(x)$ and $I_{-n}(x)$, where

$$I_n(x) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

16. If n is an integer,

$$I_n(x) = I_{-n}(x) = \frac{x^n}{2^n n!} \left\{ 1 + \frac{x^2}{2^2 \cdot 1!(n+1)} + \frac{x^4}{2^4 \cdot 2!(n+1)(n+2)} + \frac{x^6}{2^6 \cdot 3!(n+1)(n+2)(n+3)} + \dots \right\}$$

17. For $N=0$ and $n=1$, this formula becomes

$$\begin{aligned} I_0(x) &= 1 + \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} + \frac{x^6}{2^6(3!)^2} + \frac{x^8}{2^8(4!)^2} + \dots \\ I_1(x) &= \frac{x}{2} + \frac{x^3}{2^3 \cdot 1!2!} + \frac{x^5}{2^5 \cdot 2!3!} + \frac{x^7}{2^7 \cdot 3!4!} + \frac{x^9}{2^9 \cdot 4!5!} + \dots \end{aligned}$$

18. Another solution to the modified Bessel's equation is

$$K_n(x) = \begin{cases} \frac{1}{2} \pi \frac{I_{-n}(x) - I_n(x)}{\sin(n\pi)} & n \text{ not an integer} \\ \lim_{\nu \rightarrow n} \frac{1}{2} \pi \frac{I_{-\nu}(x) - I_{\nu}(x)}{\sin(\nu\pi)} & n \text{ an integer} \end{cases}$$

This function is linearly independent of $I_n(x)$ for all values of n . Thus the complete solution to the modified Bessel's equation may be written as

$$c_1 I_n(x) + c_2 I_{-n}(x) \quad n \text{ not an integer}$$

or

$$c_1 I_n(x) + c_2 K_n(x) \quad \text{any } n$$

19. The following relations hold among the various Bessel functions:

$$\begin{aligned} I_n(z) &= i^{-n} J_n(iz) \\ Y_n(iz) &= (i)^{n+1} I_n(z) - \frac{2}{\pi} i^{-n} K_n(z) \end{aligned}$$

Most of the properties of the modified Bessel function may be deduced from the known properties of $J_n(x)$ by use of these relations and those previously given.

20. Recurrence formulas

$$\begin{aligned} I_{n-1}(x) - I_{n+1}(x) &= \frac{2n}{x} I_n(x) & I_{n-1}(x) + I_{n+1}(x) &= 2I'_n(x) \\ I_{n-1}(x) - \frac{n}{x} I_n(x) &= I'_n(x) & I'_n(x) &= I_{n+1}(x) + \frac{n}{x} I_n(x) \end{aligned}$$

The Gamma Function*

Definition: $\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt \quad n > 0$

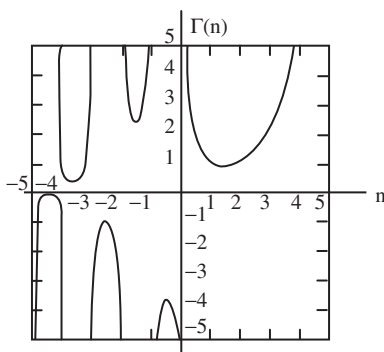
Recursion Formula: $\Gamma(n+1) = n\Gamma(n)$
 $\Gamma(n+1) = n!$, if $n = 0, 1, 2, \dots$ where $0! = 1$
 For $n < 0$ the gamma function can be defined by using

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

* From Beyer, W. H., Ed., *CRC Handbook of Mathematical Sciences*, 5th ed., CRC Press, Boca Raton, 1978, 484–485. With permission.

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

Graph:



Special Values:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(m + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2^m} \sqrt{\pi} \quad m = 1, 2, 3, \dots$$

$$\Gamma\left(-m + \frac{1}{2}\right) = \frac{(-1)^m 2^m \sqrt{\pi}}{1 \cdot 3 \cdot 5 \cdots (2m-1)} \quad m = 1, 2, 3, \dots$$

Definition:

$$\Gamma(x+1) = \lim_{k \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots k}{(x+1)(x+2) \cdots (x+k)} k^x$$

$$\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{m=1}^{\infty} \left\{ \left(1 + \frac{x}{m}\right) e^{-x/m} \right\}$$

This is an infinite product representation for the gamma function where γ is Euler's constant.

Properties:

$$\Gamma'(1) = \int_0^\infty e^{\gamma x} \ln x \, dx = -\gamma$$

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \left(\frac{1}{1} - \frac{1}{x}\right) + \left(\frac{1}{2} - \frac{1}{x+1}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{x+n-1}\right) + \cdots$$

$$\Gamma(x+1) = \sqrt{2\pi x} x^x e^{-x} \left\{ 1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51,840x^3} + \cdots \right\}$$

This is called *Stirling's asymptotic series*.

If we let $x = n$ a positive integer, then a useful approximation for $n!$ where n is large (e.g., $n > 10$) is given by *Stirling's formula*

$$n! \approx \sqrt{2\pi n} n^n e^{-n}$$

The Gamma Function*

Values of $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$; $\Gamma(n+1) = n\Gamma(n)$

n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$
1.00	1.00000	1.25	0.90640	1.50	.88623	1.75	.91906
1.01	.99433	1.26	.90440	1.51	.88659	1.76	.92137
1.02	.98884	1.27	.90250	1.52	.88704	1.77	.92376
1.03	.98355	1.28	.90072	1.53	.88757	1.78	.92623
1.04	.97844	1.29	.89904	1.54	.88818	1.79	.92877
1.05	.97350	1.30	.89747	1.55	.88887	1.80	.93138
1.06	.96874	1.31	.89600	1.56	.88964	1.81	.93408
1.07	.96415	1.32	.89464	1.57	.89049	1.82	.93685
1.08	.95973	1.33	.89338	1.58	.89142	1.83	.93969
1.09	.95546	1.34	.89222	1.59	.89243	1.84	.94261
1.10	.95135	1.35	.89115	1.60	.89352	1.85	.94561
1.11	.94740	1.36	.89018	1.61	.89468	1.86	.94869
1.12	.94359	1.37	.88931	1.62	.89592	1.87	.95184
1.13	.93993	1.38	.88854	1.63	.89724	1.88	.95507
1.14	.93642	1.39	.88785	1.64	.89864	1.89	.95838
1.15	.93304	1.40	.88726	1.65	.90012	1.90	.96177
1.16	.92980	1.41	.88676	1.66	.90167	1.91	.96523
1.17	.92670	1.42	.88636	1.67	.90330	1.92	.96877
1.18	.92373	1.43	.88604	1.68	.90500	1.93	.97240
1.19	.92089	1.44	.88581	1.69	.90678	1.94	.97610
1.20	.91817	1.45	.88566	1.70	.90864	1.95	.97988
1.21	.91558	1.46	.88560	1.71	.91057	1.96	.98374
1.22	.91311	1.47	.88563	1.72	.91258	1.97	.98768
1.23	.91075	1.48	.88575	1.73	.91466	1.98	.99171
1.24	.90852	1.49	.88595	1.74	.91683	1.99	.99581
						2.00	1.00000

* For large positive values of x , $\Gamma(x)$ approximates Stirling's asymptotic series

$$x^x e^{-x} \sqrt{\frac{2\pi}{x}} \left[1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + \cdots \right]$$

The Beta Function*

Definition: $B(m, n) = \int_0^1 t^{m-1} (1-t)^{n-1} dt$ $m > 0, n > 0$

Relationship with

Gamma Function: $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Properties:

$$B(m, n) = B(n, m)$$

$$B(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

$$B(m, n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt$$

$$B(m, n) = r^n (r+1)^m \int_0^1 \frac{t^{m-1} (1-t)^{n-1}}{(r+t)^{m+n}} dt$$

The Error Function

Definition: $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

Series: $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{1}{2!} \frac{x^5}{5} - \frac{1}{3!} \frac{x^7}{7} + \cdots \right)$

Property: $\operatorname{erf}(x) = -\operatorname{erf}(-x)$

Relationship with Normal Probability Function $f(t)$: $\int_0^x f(t) dt = \frac{1}{2} \operatorname{erf} \left(\frac{x}{\sqrt{2}} \right)$

To evaluate $\operatorname{erf}(2.3)$, one proceeds as follows: Since $\frac{x}{\sqrt{2}} = 2.3$, one finds $x = (2.3)(\sqrt{2}) = 3.25$. In the normal probability function table (page A-104), one finds the entry 0.4994 opposite the value 3.25. Thus $\operatorname{erf}(2.3) = 2(0.4994) = 0.9988$.

$$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt$$

is known as the complementary error function.

Orthogonal Polynomials*

I

Name: Legendre *Symbol:* $P_n(x)$ *Interval:* $[-1, 1]$

Differential Equation: $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$

$$y = P_n(x)$$

Explicit Expression: $P_n(x) = \frac{1}{2^n} \sum_{m=0}^{[n/2]} (-1)^m \binom{n}{m} \binom{2n-2m}{n} x^{n-2m}$

Recurrence Relation: $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

Weight: 1 *Standardization:* $P_n(1) = 1$

Norm: $\int_{-1}^{+1} [P_n(x)]^2 dx = \frac{2}{2n+1}$

Rodrigues' Formula: $P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} \{(1-x^2)^n\}$

Generating Function: $R^{-1} = \sum_{n=0}^{\infty} P_n(x) z^n, -1 < x < 1, \quad |z| < 1,$

$$R = \sqrt{1 - 2xz + z^2}$$

Inequality: $|P_n(x)| \leq 1, \quad -1 \leq x \leq 1.$

II

Name: Tschebysheff, First Kind *Symbol:* $T_n(x)$ *Interval:* $[-1, 1]$

Differential Equation: $(1 - x^2)y - xy' + n^2y = 0$

$$y = T_n(x)$$

Explicit Expression: $\frac{n}{2} \sum_{m=0}^{[n/2]} (-1)^m \frac{(n-m-1)!}{m!(n-2m)!} (2x)^{n-2m} = \cos(n \arccos x) = T_n(x)$

Recurrence Relation: $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

Weight: $(1 - x^2)^{-1/2}$ *Standardization:* $T_n(1) = 1$

Norm: $\int_{-1}^{+1} (1 - x^2)^{-1/2} [T_n(x)]^2 dx = \begin{cases} \pi/2, & n \neq 0 \\ \pi, & n = 0 \end{cases}$

Rodrigues' Formula: $\frac{(-1)^n (1 - x^2)^{1/2} \sqrt{\pi}}{2^{n+1} \Gamma(n + \frac{1}{2})} \frac{d^n}{dx^n} \{(1 - x^2)^{n-(1/2)}\} = T_n(x)$

Generating Function: $\frac{1 - xz}{1 - 2xz - z^2} = \sum_{n=0}^{\infty} T_n(x) z^n, \quad -1 < x < 1, \quad |z| < 1$

Inequality: $|T_n(x)| \leq 1, \quad -1 \leq x \leq 1.$

III

Name: Tschebysheff, Second Kind *Symbol:* $U_n(x)$ *Interval:* $[-1, 1]$

Differential Equation: $(1 - x^2)y'' - 3xy' + n(n+2)y = 0$

$$y = U_n(x)$$

Explicit Expression: $U_n(x) = \sum_{m=0}^{[n/2]} (-1)^m \frac{(m-n)!}{m!(n-2m)!} (2x)^{n-2m}$

$$U_n(\cos \theta) = \frac{\sin[(n+1)\theta]}{\sin \theta}$$

* From Beyer, W. H., Ed., *CRC Handbook of Mathematical Sciences*, 5th ed., CRC Press, Boca Raton, 1978, 557–560. With permission.

Recurrence Relation: $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$

Weight: $(1-x^2)^{1/2}$ *Standardization:* $U_n(1) = n+1$

Norm: $\int_{-1}^{+1} (1-x^2)^{1/2} [U_n(x)]^2 dx = \frac{\pi}{2}$

Rodrigues' Formula: $U_n(x) = \frac{(-1)^n (n+1) \sqrt{\pi}}{(1-x^2)^{1/2} 2^{n+1} \Gamma(n+\frac{3}{2})} \frac{d^n}{dx^n} \{(1-x^2)^{n+(1/2)}\}$

Generating Function: $\frac{1}{1-2xz+z^2} = \sum_{n=0}^{\infty} U_n(x) z^n, -1 < x < 1, |z| < 1$

Inequality: $|U_n(x)| \leq n+1, -1 \leq x \leq 1.$

IV

Name: Jacobi *Symbol:* $P_n^{(\alpha, \beta)}(x)$ *Interval:* $[-1, 1]$

Differential Equation:

$$(1-x^2)y'' + [\beta - \alpha - (\alpha + \beta + 2)x]y' + n(n + \alpha + \beta + 1)y = 0$$

$$y = P_n^{(\alpha, \beta)}(x)$$

Explicit Expression: $P_n^{(\alpha, \beta)}(x) = \frac{1}{2^n} \sum_{m=0}^n \binom{n+\alpha}{m} \binom{n+\beta}{n-m} (x-1)^{n-m} (x+1)^m$

Recurrence Relation:

$$2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha, \beta)}(x)$$

$$= (2n+\alpha+\beta+1)[(\alpha^2 - \beta^2) + (2n+\alpha+\beta+2)$$

$$\times (2n+\alpha+\beta)x]P_n^{(\alpha, \beta)}(x)$$

$$- 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{(\alpha, \beta)}(x)$$

Weight: $(1-x)^\alpha(1+x)^\beta; \alpha, \beta > -1$ *Standardization:* $P_n^{(\alpha, \beta)}(x) = \binom{n+\alpha}{n}$

Norm: $\int_{-1}^{+1} (1-x)^\alpha(1+x)^\beta [P_n^{(\alpha, \beta)}(x)]^2 dx = \frac{2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n! \Gamma(n+\alpha+\beta+1)}$

Rodrigues' Formula: $P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n! (1-x)^\alpha (1+x)^\beta} \frac{d^n}{dx^n} \{(1-x)^{n+\alpha} (1+x)^{n+\beta}\}$

Generating Function: $R^{-1}(1-z+R)^{-\alpha}(1+z+R)^{-\beta} = \sum_{n=0}^{\infty} 2^{-\alpha-\beta} P_n^{(\alpha, \beta)}(x) z^n,$

$$R = \sqrt{1-2xz+z^2}, \quad |z| < 1$$

Inequality: $\max_{-1 \leq x \leq 1} |P_n^{(\alpha, \beta)}(x)| = \begin{cases} \binom{n+q}{n} \sim n^q & \text{if } q = \max(\alpha, \beta) \geq -\frac{1}{2} \\ |P_n^{(\alpha, \beta)}(x')| \sim n^{-1/2} & \text{if } q < -\frac{1}{2} \\ x' \text{ is one of the two maximum points nearest} \\ \frac{\beta - \alpha}{\alpha + \beta + 1} \end{cases}$

V

Name: Generalized Laguerre *Symbol:* $L_n^{(\alpha)}(x)$ *Interval:* $[0, \infty]$

Differential Equation: $xy'' + (\alpha+1-x)y' + ny = 0$

$$y = L_n^{(\alpha)}(x)$$

Explicit Expression: $L_n^{(\alpha)}(x) = \sum_{m=0}^n (-1)^m \binom{n+\alpha}{n-m} \frac{1}{m!} x^m$

Recurrence Relation: $(n+1)L_n^{(\alpha)} + 1(x) = [(2n+\alpha+1)-x]L_n^{(\alpha)}(x) - (n+\alpha)L_n^{(\alpha)} - 1(x)$

Weight: $x^\alpha e^{-x}, \alpha > -1$ *Standardization:* $L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} x^n + \dots$

Norm: $\int_0^\infty x^\alpha e^{-x} [L_n^{(\alpha)}(x)]^2 dx = \frac{\Gamma(n+\alpha+1)}{n!}$

Rodrigues' Formula: $L_n^{(\alpha)}(x) = \frac{1}{n! x^\alpha e^{-x}} \frac{d^n}{dx^n} \{x^{n+\alpha} e^{-x}\}$

Generating Function: $(1-z)^{-\alpha-1} \exp\left(\frac{xz}{z-1}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) z^n$

Inequality: $|L_n^{(\alpha)}(x)| \leq \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} e^{x/2}; \quad \begin{matrix} x \geq 0 \\ \alpha > 0 \end{matrix}$

$|L_n^{(\alpha)}(x)| \leq \left[2 - \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)}\right] e^{x/2}; \quad \begin{matrix} x \geq 0 \\ -1 < \alpha < 0 \end{matrix}$

Orthogonal Polynomials

Name: Hermite *Symbol:* $H_n(x)$ *Interval:* $[-\infty, \infty]$

Differential Equation: $y'' - 2xy' + 2ny = 0$

Explicit Expression: $H_n(x) = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m n! (2x)^{n-2m}}{m!(n-2m)!}$

Recurrence Relation: $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$

Weight: e^{-x^2} *Standardization:* $H_n(1) = 2^n x^n + \dots$

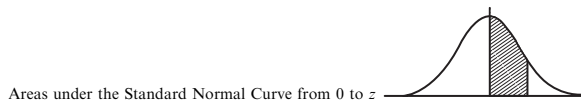
Norm: $\int_{-\infty}^{\infty} e^{-x^2} [H_n(x)]^2 dx = 2^n n! \sqrt{\pi}$

Rodrigues' Formula: $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2})$

Generating Function: $e^{-z^2+2zx} = \sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!}$

Inequality: $|H_n(x)| < e^{x^2/2} k 2^{n/2} \sqrt{n!} \quad k \approx 1.086435$

NORMAL PROBABILITY FUNCTION



z	0	1	2	3	4	5	6	7	8	9
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0754
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2258	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549
0.7	.2580	.2612	.2652	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2996	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000

$F(z)$ below refers to area under Standard Normal Curve from $-\infty$ to z

z	1.282	1.645	1.960	2.326	2.576	3.090
$F(z)$	.90	.95	.975	.99	.995	.999
$2[1 - F(z)]$	.20	.10	.05	.02	.01	.002

PERCENTAGE POINTS, STUDENT'S *t*-DISTRIBUTION

This table gives values of *t* such that

$$F(t) = \int_{-\infty}^t \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}} dx$$

for *n*, the number of degrees of freedom, equal to 1, 2, . . . , 30, 40, 60, 120, ∞; and for *F*(*t*) = 0.60, 0.75, 0.90, 0.95, 0.975, 0.99, 0.995, and 0.9995. The *t*-distribution is symmetrical, so that *F*(−*t*) = 1 − *F*(*t*)

<i>n</i> \ <i>F</i>	.60	.75	.90	.95	.975	.99	.995	.9995
1	.325	1.000	3.078	6.314	12.706	31.821	63.657	636.619
2	.289	.816	1.886	2.920	4.303	6.965	9.925	31.598
3	.277	.765	1.638	2.353	3.182	4.541	5.841	12.924
4	.271	.741	1.533	2.132	2.776	3.747	4.604	8.610
5	.267	.727	1.476	2.015	2.571	3.365	4.032	6.869
6	.265	.718	1.440	1.943	2.447	3.143	3.707	5.959
7	.263	.711	1.415	1.895	2.365	2.998	3.499	5.408
8	.262	.706	1.397	1.860	2.306	2.896	3.355	5.041
9	.261	.703	1.383	1.833	2.262	2.821	3.250	4.781
10	.260	.700	1.372	1.812	2.228	2.764	3.169	4.587
11	.260	.697	1.363	1.796	2.201	2.718	3.106	4.437
12	.259	.695	1.356	1.782	2.179	2.681	3.055	4.318
13	.259	.694	1.350	1.771	2.160	2.650	3.012	4.221
14	.258	.692	1.345	1.761	2.145	2.624	2.977	4.140
15	.258	.691	1.341	1.753	2.131	2.602	2.947	4.073
16	.258	.690	1.337	1.746	2.120	2.583	2.921	4.015
17	.257	.689	1.333	1.740	2.110	2.567	2.898	3.965
18	.257	.688	1.330	1.734	2.101	2.552	2.878	3.922
19	.257	.688	1.328	1.729	2.093	2.539	2.861	3.883
20	.257	.687	1.325	1.725	2.086	2.528	2.845	3.850
21	.257	.686	1.323	1.721	2.080	2.518	2.831	3.819
22	.256	.686	1.321	1.717	2.074	2.508	2.819	3.792
23	.256	.685	1.319	1.714	2.069	2.500	2.807	3.767
24	.256	.685	1.318	1.711	2.064	2.492	2.797	3.745
25	.256	.684	1.316	1.708	2.060	2.485	2.787	3.725
26	.256	.684	1.315	1.706	2.056	2.479	2.779	3.707
27	.256	.684	1.314	1.703	2.052	2.473	2.771	3.690
28	.256	.683	1.313	1.701	2.048	2.467	2.763	3.674
29	.256	.683	1.311	1.699	2.045	2.462	2.756	3.659
30	.256	.683	1.310	1.697	2.042	2.457	2.750	3.646
40	.255	.681	1.303	1.684	2.021	2.423	2.704	3.551
60	.254	.679	1.296	1.671	2.000	2.390	2.660	3.460
120	.254	.677	1.289	1.658	1.980	2.358	2.617	3.373
∞	.253	.674	1.282	1.645	1.960	2.326	2.576	3.291

* This table is abridged from the “Statistical Tables” of R. A. Fisher and Frank Yates published by Oliver & Boyd. Ltd., Edinburgh and London, 1938. It is here published with the kind permission of the authors and their publishers.

PERCENTAGE POINTS, CHI-SQUARE DISTRIBUTION

This table gives values of χ^2 such that

$$F(\chi)^2 = \int_0^{\chi^2} \frac{1}{2^{\frac{n}{2}}\Gamma(n/2)} x^{(n-2)/2} e^{-(x/2)} dx$$

for n , the number of degrees of freedom, equal to $1, 2, \dots, 30$. For $n > 30$, a normal approximation is quite accurate. The expression $\sqrt{2\chi^2} - \sqrt{2n-1}$ is approximately normally distributed as the standard normal distribution. Thus χ_α^2 , the α -point of the distribution, may be computed by the formula

$$\chi_\alpha^2 = \frac{1}{2}[x_\alpha + \sqrt{2n-1}]^2,$$

where x_α is the α -point of the cumulative normal distribution. For even values of n , $F(\chi^2)$ can be written as

$$1 - F(\chi^2) = \sum_{x=0}^{x'-1} \frac{e^{-\lambda} \lambda^x}{x!}$$

with $\lambda = \frac{1}{2}\chi^2$ and $x' = \frac{1}{2}n$. Thus the cumulative Chi-Square distribution is related to the cumulative Poisson distribution.

Another approximate formula for large n

$$\chi^2_{\alpha} = n \left(1 - \frac{2}{9n} + z_{\alpha} \sqrt{\frac{2}{9n}} \right)^3$$

n = degrees of freedom
 z_{α} = the normal deviate (the value of x for which $F(x)$ = the desired percentile).

x	1.282	1.645	1.960	2.326	2.576	3.090
$F(x)$	.90	.95	.975	.99	.995	.999

$\chi^2_{.99} = 60[1 - 0.00370 + 2.326(0.06086)]^3 = 88.4$ is the 99th percentile for 60 degrees of freedom.

$$F(\chi^2) = \int_0^{\chi^2} \frac{1}{2^{n/2} \Gamma(\frac{n}{2})} x^{(n-2)/2} e^{-x/2} dx$$

$n \backslash F$	.005	.010	.025	.050	.100	.250	.500	.750	.900	.950	.975	.990	.995
1	.0000393	.000157	.000982	.00393	.0158	.102	.455	1.32	2.71	3.84	5.02	6.63	7.88
2	.0100	.0201	.0506	.103	.211	.575	1.39	2.77	4.61	5.99	7.38	9.21	10.6
3	.0717	.115	.216	.352	.584	1.21	2.37	4.11	6.25	7.81	9.35	11.3	12.8
4	.207	.297	.484	.711	1.06	1.92	3.36	5.39	7.78	9.49	11.1	13.3	14.9
5	.412	.554	.831	1.15	1.61	2.67	4.35	6.63	9.24	11.1	12.8	15.1	16.7
6	.676	.872	1.24	1.64	2.20	3.45	5.35	7.84	10.6	12.6	14.4	16.8	18.5
7	.989	1.24	1.69	2.17	2.83	4.25	6.35	9.04	12.0	14.1	16.0	18.5	20.3
8	1.34	1.65	2.18	2.73	3.49	5.07	7.34	10.2	13.4	15.5	17.5	20.1	22.0
9	1.73	2.09	2.70	3.33	4.17	5.90	8.34	11.4	14.7	16.9	19.0	21.7	23.6
10	2.16	2.56	3.25	3.94	4.87	6.74	9.34	12.5	16.0	18.3	20.5	23.2	25.2
11	2.60	3.05	3.82	4.57	5.58	7.58	10.3	13.7	17.3	19.7	21.9	24.7	26.8
12	3.07	3.57	4.40	5.23	6.30	8.44	11.3	14.8	18.5	21.0	23.3	26.2	28.3
13	3.57	4.11	5.01	5.89	7.04	9.30	12.3	16.0	19.8	22.4	24.7	27.7	29.8
14	4.07	4.66	5.63	6.57	7.79	10.2	13.3	17.1	21.1	23.7	26.1	29.1	31.3
15	4.60	5.23	6.26	7.26	8.55	11.0	14.3	18.2	22.3	25.0	27.5	30.6	32.8
16	5.14	5.81	6.91	7.96	9.31	11.9	15.3	19.4	23.5	26.3	28.8	32.0	34.3
17	5.70	6.41	7.56	8.67	10.1	12.8	16.3	20.5	24.8	27.6	30.2	33.4	35.7
18	6.26	7.01	8.23	9.39	10.9	13.7	17.3	21.6	26.0	28.9	31.5	34.8	37.2
19	6.84	7.63	8.91	10.1	11.7	14.6	18.3	22.7	27.2	30.1	32.9	36.2	38.6
20	7.43	8.26	9.59	10.9	12.4	15.5	19.3	23.8	28.4	31.4	34.2	37.6	40.0
21	8.03	8.90	10.3	11.6	13.2	16.3	20.3	24.9	29.6	32.7	35.5	38.9	41.4
22	8.64	9.54	11.0	12.3	14.0	17.2	21.3	26.0	30.8	33.9	36.8	40.3	42.8
23	9.26	10.2	11.7	13.1	14.8	18.1	22.3	27.1	32.0	35.2	38.1	41.6	44.2
24	9.89	10.9	12.4	13.8	15.7	19.0	23.3	28.2	33.2	36.4	39.4	43.0	45.6
25	10.5	11.5	13.1	14.6	16.5	19.9	24.3	29.3	34.4	37.7	40.6	44.3	46.9
26	11.2	12.2	13.8	15.4	17.3	20.8	25.3	30.4	35.6	38.9	41.9	45.6	48.3
27	11.8	12.9	14.6	16.2	18.1	21.7	26.3	31.5	36.7	40.1	43.2	47.0	49.6
28	12.5	13.6	15.3	16.9	18.9	22.7	27.3	32.6	37.9	41.3	44.5	48.3	51.0
29	13.1	14.3	16.0	17.7	19.8	23.6	28.3	33.7	39.1	42.6	45.7	49.6	52.3
30	13.8	15.0	16.8	18.5	20.6	24.5	29.3	34.8	40.3	43.8	47.0	50.9	53.7

PERCENTAGE POINTS, F -DISTRIBUTION

This table gives values of F such that

$$F(F) = \int_0^F \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} m^{m/2} n^{n/2} x^{(m-2)/2} (n+mx)^{-(m+n)/2} dx$$

for selected values of m , the number of degrees of freedom of the numerator of F ; and for selected values of n , the number of degrees of freedom of the denominator of F . The table also provides values corresponding to $F(F) = .10, .05, .025, .01, .005, .001$ since $F_{1-\alpha}$ for m and n degrees of freedom is the reciprocal of F_α for n and m degrees of freedom. Thus

$$F_{.05}(4, 7) = \frac{1}{F_{.95}(7, 4)} = \frac{1}{6.09} = .164$$

$$F(F) = \int_0^F \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} m^{m/2} n^{n/2} x^{m/2-1} (n+mx)^{-(m+n)/2} dx = .90$$

$\frac{n}{m}$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	59.86	60.19	60.71	61.22	61.74	62.00	62.26	62.53	62.79	63.06	63.33
2	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.41	9.42	9.44	9.45	9.46	9.47	9.47	9.48	9.49
3	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.22	5.20	5.18	5.18	5.17	5.16	5.15	5.14	5.13
4	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.90	3.87	3.84	3.83	3.82	3.80	3.79	3.78	3.76
5	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.27	3.24	3.21	3.19	3.17	3.16	3.14	3.12	3.10
6	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.90	2.87	2.84	2.82	2.80	2.78	2.76	2.74	2.72
7	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.67	2.63	2.59	2.58	2.56	2.54	2.51	2.49	2.47
8	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.50	2.46	2.42	2.40	2.38	2.36	2.34	2.32	2.29
9	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.38	2.34	2.30	2.28	2.25	2.23	2.21	2.18	2.16
10	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.28	2.24	2.20	2.18	2.16	2.13	2.11	2.08	2.06
11	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.21	2.17	2.12	2.10	2.08	2.05	2.03	2.00	1.97
12	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19	2.15	2.10	2.06	2.04	2.01	1.99	1.96	1.93	1.90
13	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14	2.10	2.05	2.01	1.98	1.96	1.93	1.90	1.88	1.85
14	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10	2.05	2.01	1.96	1.94	1.91	1.89	1.86	1.83	1.80
15	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06	2.02	1.97	1.92	1.90	1.87	1.85	1.82	1.79	1.76
16	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06	2.03	1.99	1.94	1.89	1.87	1.84	1.81	1.78	1.75	1.72
17	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03	2.00	1.96	1.91	1.86	1.84	1.81	1.78	1.75	1.72	1.69
18	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00	1.98	1.93	1.89	1.84	1.81	1.78	1.75	1.72	1.69	1.66
19	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96	1.91	1.86	1.81	1.79	1.76	1.73	1.70	1.67	1.63
20	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96	1.94	1.89	1.84	1.79	1.77	1.74	1.71	1.68	1.64	1.61
21	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95	1.92	1.87	1.83	1.78	1.75	1.72	1.69	1.66	1.62	1.59
22	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90	1.86	1.81	1.76	1.73	1.70	1.67	1.64	1.60	1.57
23	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92	1.89	1.84	1.80	1.74	1.72	1.69	1.66	1.62	1.59	1.55
24	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88	1.83	1.78	1.73	1.70	1.67	1.64	1.61	1.57	1.53
25	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89	1.87	1.82	1.77	1.72	1.69	1.66	1.63	1.59	1.56	1.52
26	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86	1.81	1.76	1.71	1.68	1.65	1.61	1.58	1.54	1.50
27	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87	1.85	1.80	1.75	1.70	1.67	1.64	1.60	1.57	1.53	1.49
28	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84	1.79	1.74	1.69	1.66	1.63	1.59	1.56	1.52	1.48
29	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86	1.83	1.78	1.73	1.68	1.65	1.62	1.58	1.55	1.51	1.47
30	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85	1.82	1.77	1.72	1.67	1.64	1.61	1.57	1.54	1.50	1.46
40	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79	1.76	1.71	1.66	1.61	1.57	1.54	1.51	1.47	1.42	1.38
60	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.66	1.60	1.54	1.51	1.48	1.44	1.40	1.35	1.29
120	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68	1.65	1.60	1.55	1.48	1.45	1.41	1.37	1.32	1.26	1.19
∞	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63	1.60	1.55	1.49	1.42	1.38	1.34	1.30	1.24	1.17	1.00

$F = \frac{s_1^2}{s_2^2} = \frac{S_1}{m} \bigg/ \frac{S_2}{n}$, where $s_1^2 = S_1/m$ and $s_2^2 = S_2/n$ are independent mean squares estimating a common variance σ^2 and based on m and n degrees of freedom, respectively.

$$F(F) = \int_0^F \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} m^{m/2} n^{n/2} x^{m/2-1} (n+mx)^{-(m+n)/2} dx = .95$$

$\frac{n}{m}$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

$$F(F) = \int_0^F \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} m^{m/2} n^{n/2} x^{m/2-1} (n+mx)^{-(m+n)/2} dx = .975$$

$\frac{m}{n}$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	647.8	799.5	864.2	899.6	921.8	937.1	948.2	956.7	963.3	968.6	976.7	984.9	993.1	997.2	1001	1006	1010	1014	1018
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.43	39.45	39.46	39.46	39.47	39.48	39.49	39.50
3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.34	14.25	14.17	14.12	14.08	14.04	13.99	13.95	13.90
4	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66	8.56	8.51	8.46	8.41	8.36	8.31	8.26
5	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43	6.33	6.28	6.23	6.18	6.12	6.07	6.02
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27	5.17	5.12	5.07	5.01	4.96	4.90	4.85
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57	4.47	4.42	4.36	4.31	4.25	4.20	4.14
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10	4.00	3.95	3.89	3.84	3.78	3.73	3.67
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77	3.67	3.61	3.56	3.51	3.45	3.39	3.33
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52	3.42	3.37	3.31	3.26	3.20	3.14	3.08
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.43	3.33	3.23	3.17	3.12	3.06	3.00	2.94	2.88
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18	3.07	3.02	2.96	2.91	2.85	2.79	2.72
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.15	3.05	2.95	2.89	2.84	2.78	2.72	2.66	2.60
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.05	2.95	2.84	2.79	2.73	2.67	2.61	2.55	2.49
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86	2.76	2.70	2.64	2.59	2.52	2.46	2.40
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.89	2.79	2.68	2.63	2.57	2.51	2.45	2.38	2.32
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.82	2.72	2.62	2.56	2.50	2.44	2.38	2.32	2.25
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.77	2.67	2.56	2.50	2.44	2.38	2.32	2.26	2.19
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.72	2.62	2.51	2.45	2.39	2.33	2.27	2.20	2.13
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.57	2.46	2.41	2.35	2.29	2.22	2.16	2.09
21	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.64	2.53	2.42	2.37	2.31	2.25	2.18	2.11	2.04
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.60	2.50	2.39	2.33	2.27	2.21	2.14	2.08	2.00
23	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.57	2.47	2.36	2.30	2.24	2.18	2.11	2.04	1.97
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.54	2.44	2.33	2.27	2.21	2.15	2.08	2.01	1.94
25	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.51	2.41	2.30	2.24	2.18	2.12	2.05	1.98	1.91
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.49	2.39	2.28	2.22	2.16	2.09	2.03	1.95	1.88
27	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63	2.57	2.47	2.36	2.25	2.19	2.13	2.03	2.00	1.93	1.85
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.45	2.34	2.23	2.17	2.11	2.05	1.98	1.91	1.83
29	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59	2.53	2.43	2.32	2.21	2.15	2.09	2.03	1.96	1.89	1.81
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.41	2.31	2.20	2.14	2.07	2.01	1.94	1.87	1.79
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.29	2.18	2.07	2.01	1.94	1.88	1.80	1.72	1.64
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.17	2.06	1.94	1.88	1.82	1.74	1.67	1.58	1.48
120	5.15	3.80	3.23	2.89	2.67	2.52	2.39	2.30	2.22	2.16	2.05	1.94	1.82	1.76	1.69	1.64	1.53	1.43	1.31
∞	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.94	1.83	1.71	1.64	1.57	1.48	1.39	1.24	1.00

$F = \frac{s_1^2}{s_2^2} = \frac{S_1}{m} \bigg/ \frac{S_2}{n}$, where $s_1^2 = S_1/m$ and $s_2^2 = S_2/n$ are independent mean squares estimating a common variance σ^2 and based on m and n degrees of freedom, respectively.

$$F(F) = \int_0^F \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} m^{m/2} n^{n/2} x^{m/2-1} (n+mx)^{-(m+n)/2} dx = .99$$

$\frac{m}{n}$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	4052	4999.5	5403	5625	5764	5859	5928	5982	6022	6056	6106	6157	6209	6235	6261	6287	6313	6339	6366
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48	99.49	99.50
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32	26.22	26.13
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36	2.27	2.17
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	2.96	2.81	2.66	2.58	2.50	2.42	2.33	2.23	2.13
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29	2.20	2.10
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.90	2.75	2.60	2.52	2.44	2.35	2.26	2.17	2.06
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23	2.14	2.03
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00

$$F(F) = \int_0^F \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} m^{m/2} n^{n/2} x^{m/2-1} (n+mx)^{-(m+n)/2} dx = .995$$

$\begin{smallmatrix} m \\ n \end{smallmatrix}$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	16211	2000	21615	22500	23056	23437	23715	23925	24091	24224	24426	24630	24836	24940	25044	25148	25253	25359	25465
2	198.5	199.0	199.2	199.2	199.3	199.3	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.5	199.5	199.5	199.5	199.5	199.5
3	55.55	49.80	47.47	46.19	45.39	44.84	44.43	44.13	43.88	43.69	43.39	43.08	42.78	42.62	42.47	42.31	42.15	41.99	41.83
4	31.33	26.28	24.26	23.15	22.46	21.97	21.62	21.35	21.14	20.97	20.70	20.44	20.17	20.03	19.89	19.75	19.61	19.47	19.32
5	22.78	18.31	16.53	15.56	14.94	14.51	14.20	13.96	13.77	13.62	13.38	13.15	12.90	12.78	12.66	12.53	12.40	12.27	12.14
6	18.63	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.39	10.25	10.03	9.81	9.59	9.47	9.36	9.24	9.12	9.00	8.88
7	16.24	12.40	10.88	10.05	9.52	9.16	8.89	8.68	8.51	8.38	8.18	7.97	7.75	7.65	7.53	7.42	7.31	7.19	7.08
8	14.69	11.04	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	7.01	6.81	6.61	6.50	6.40	6.29	6.18	6.06	5.95
9	13.61	10.11	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.23	6.03	5.83	5.73	5.62	5.52	5.41	5.30	5.19
10	12.83	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.66	5.47	5.27	5.17	5.07	4.97	4.86	4.75	4.64
11	12.23	8.91	7.60	6.88	6.42	6.10	5.86	5.68	5.54	5.42	5.24	5.05	4.86	4.76	4.65	4.55	4.44	4.34	4.23
12	11.75	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.91	4.72	4.53	4.43	4.33	4.23	4.12	4.01	3.90
13	11.37	8.19	6.93	6.23	5.79	5.48	5.25	5.08	4.94	4.82	4.64	4.46	4.27	4.17	4.07	3.97	3.87	3.76	3.65
14	11.06	7.92	6.68	6.00	5.56	5.26	5.03	4.86	4.72	4.60	4.43	4.25	4.06	3.96	3.86	3.76	3.66	3.55	3.44
15	10.80	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.25	4.07	3.88	3.79	3.69	3.58	3.48	3.37	3.26
16	10.58	7.51	6.30	5.64	5.21	4.91	4.69	4.52	4.38	4.27	4.10	3.92	3.73	3.64	3.54	3.44	3.33	3.22	3.11
17	10.38	7.35	6.16	5.50	5.07	4.78	4.56	4.39	4.25	4.14	3.97	3.79	3.61	3.51	3.41	3.31	3.21	3.10	2.98
18	10.22	7.21	6.03	5.37	4.96	4.66	4.44	4.28	4.14	4.03	3.86	3.68	3.50	3.40	3.30	3.20	3.10	2.99	2.87
19	10.07	7.09	5.92	5.27	4.85	4.56	4.34	4.18	4.04	3.93	3.76	3.59	3.40	3.31	3.21	3.11	3.00	2.89	2.78
20	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96	3.85	3.68	3.50	3.32	3.22	3.12	3.02	2.92	2.81	2.69
21	9.83	6.89	5.73	5.09	4.68	4.39	4.18	4.01	3.88	3.77	3.60	3.43	3.24	3.15	3.05	2.95	2.84	2.73	2.61
22	9.73	6.81	5.65	5.02	4.61	4.32	4.11	3.94	3.81	3.70	3.54	3.36	3.18	3.08	2.98	2.88	2.77	2.66	2.55
23	9.63	6.73	5.58	4.95	4.54	4.26	4.05	3.88	3.75	3.64	3.47	3.30	3.12	3.02	2.92	2.82	2.71	2.60	2.48
24	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69	3.59	3.42	3.25	3.06	2.97	2.87	2.77	2.66	2.55	2.43
25	9.48	6.60	5.46	4.84	4.43	4.15	3.94	3.78	3.64	3.54	3.37	3.20	3.01	2.92	2.82	2.72	2.61	2.50	2.38
26	9.41	6.54	5.41	4.79	4.38	4.10	3.89	3.73	3.60	3.49	3.33	3.15	2.97	2.87	2.77	2.67	2.56	2.45	2.33
27	9.34	6.49	5.36	4.74	4.34	4.06	3.85	3.69	3.56	3.45	3.28	3.11	2.93	2.83	2.73	2.63	2.52	2.41	2.29
28	9.28	6.44	5.32	4.70	4.30	4.02	3.81	3.65	3.52	3.41	3.25	3.07	2.89	2.79	2.69	2.59	2.48	2.37	2.25
29	9.23	6.40	5.28	4.66	4.26	3.98	3.77	3.61	3.48	3.38	3.21	3.04	2.86	2.76	2.66	2.56	2.45	2.33	2.24
30	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.34	3.18	3.01	2.82	2.73	2.63	2.52	2.42	2.30	2.18
40	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22	3.12	2.95	2.78	2.60	2.50	2.40	2.30	2.18	2.06	1.93
60	8.49	5.79	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.90	2.74	2.57	2.39	2.29	2.19	2.08	1.96	1.83	1.69
120	8.18	5.54	4.50	3.92	3.55	3.28	3.09	2.93	2.81	2.71	2.54	2.37	2.19	2.09	1.98	1.87	1.75	1.61	1.43
∞	7.88	5.30	4.28	3.72	3.35	3.09	2.90	2.74	2.62	2.52	2.36	2.19	2.00	1.90	1.79	1.67	1.53	1.36	1.00

$F = \frac{s_1^2}{s_2^2} = \frac{S_1}{m} \bigg/ \frac{S_2}{n}$, where $s_1^2 = S_1/m$ and $s_2^2 = S_2/n$ are independent mean squares estimating a common variance σ^2 and based on m and n degrees of freedom, respectively.

$$F(F) = \int_0^F \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} m^{m/2} n^{n/2} x^{m/2-1} (n+mx)^{-(m+n)/2} dx = .999$$

$n \backslash m$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	4053*	5000*	5404*	5625*	5764*	5859*	5929*	5981*	6023*	6056*	6107*	6158*	6209*	6235*	6261*	6287*	6313*	6340*	6366*
2	998.5	999.0	999.2	999.2	999.3	999.3	999.4	999.4	999.4	999.4	999.4	999.4	999.4	999.5	999.5	999.5	999.5	999.5	999.5
3	167.0	148.5	141.1	137.1	134.6	132.8	131.6	130.6	129.9	129.2	128.3	127.4	126.4	125.9	125.4	125.0	124.5	124.0	123.5
4	74.14	61.25	56.18	53.44	51.71	50.53	49.66	49.00	48.47	48.05	47.41	46.76	46.10	45.77	45.43	45.09	44.75	44.40	44.05
5	47.18	37.12	33.20	31.09	29.75	28.84	28.16	27.64	27.24	26.92	26.42	25.91	25.39	25.14	24.87	24.60	24.33	24.06	23.79
6	35.51	27.00	23.70	21.92	20.81	20.03	19.46	19.03	18.69	18.41	17.99	17.56	17.12	16.89	16.67	16.44	16.21	15.99	15.75
7	29.25	21.69	18.77	17.19	16.21	15.52	15.02	14.63	14.33	14.08	13.71	13.32	12.93	12.73	12.53	12.33	12.12	11.91	11.70
8	25.42	18.49	15.83	14.39	13.49	12.86	12.40	12.04	11.77	11.54	11.19	10.84	10.48	10.30	10.11	9.92	9.73	9.53	9.33
9	22.86	16.39	13.90	12.56	11.71	11.13	10.70	10.37	10.11	9.89	9.57	9.24	8.90	8.72	8.55	8.37	8.19	8.00	7.81
10	21.04	14.91	12.55	11.28	10.48	9.92	9.52	9.20	8.96	8.75	8.45	8.13	7.80	7.64	7.47	7.30	7.12	6.94	6.76
11	19.69	13.81	11.56	10.35	9.58	9.05	8.66	8.35	8.12	7.92	7.63	7.32	7.01	6.85	6.68	6.52	6.35	6.17	6.00
12	18.64	12.97	10.80	9.63	8.89	8.38	8.00	7.71	7.48	7.29	7.00	6.71	6.40	6.25	6.09	5.93	5.76	5.59	5.42
13	17.81	12.31	10.21	9.07	8.35	7.86	7.49	7.21	6.98	6.80	6.52	6.23	5.93	5.78	5.63	5.47	5.30	5.14	4.97
14	17.14	11.78	9.73	8.62	7.92	7.43	7.08	6.80	6.58	6.40	6.13	5.85	5.56	5.41	5.25	5.10	4.94	4.77	4.60
15	16.59	11.34	9.34	8.25	7.57	7.09	6.74	6.47	6.26	6.08	5.81	5.54	5.25	5.10	4.95	4.80	4.64	4.47	4.31
16	16.12	10.97	9.00	7.94	7.27	6.81	6.46	6.19	5.98	5.81	5.55	5.27	4.99	4.85	4.70	4.54	4.39	4.23	4.06
17	15.72	10.66	8.73	7.68	7.02	6.56	6.22	5.96	5.75	5.58	5.32	5.05	4.78	4.63	4.48	4.33	4.18	4.02	3.85
18	15.38	10.39	8.49	7.46	6.81	6.35	6.02	5.76	5.56	5.39	5.13	4.87	4.59	4.45	4.30	4.15	4.00	3.84	3.67
19	15.08	10.16	8.28	7.26	6.62	6.18	5.85	5.59	5.39	5.22	4.97	4.70	4.43	4.29	4.14	3.99	3.84	3.68	3.51
20	14.82	9.95	8.10	7.10	6.46	6.02	5.69	5.44	5.24	5.08	4.82	4.56	4.29	4.15	4.00	3.86	3.70	3.54	3.38
21	14.59	9.77	7.94	6.95	6.32	5.88	5.56	5.31	5.11	4.95	4.70	4.44	4.17	4.03	3.88	3.74	3.58	3.42	3.26
22	14.38	9.61	7.80	6.81	6.19	5.76	5.44	5.19	4.99	4.83	4.58	4.33	4.06	3.92	3.78	3.63	3.48	3.32	3.15
23	14.19	9.47	7.67	6.69	6.08	5.65	5.33	5.09	4.89	4.73	4.48	4.23	3.96	3.82	3.68	3.53	3.38	3.22	3.05
24	14.03	9.34	7.55	6.59	5.98	5.55	5.23	4.99	4.80	4.64	4.39	4.14	3.87	3.74	3.59	3.45	3.29	3.14	2.97
25	13.88	9.22	7.45	6.49	5.88	5.46	5.15	4.91	4.71	4.56	4.31	4.06	3.79	3.66	3.52	3.37	3.22	3.06	2.89
26	13.74	9.12	7.36	6.41	5.80	5.38	5.07	4.83	4.64	4.48	4.24	3.99	3.72	3.59	3.44	3.30	3.15	2.99	2.82
27	13.61	9.02	7.27	6.33	5.73	5.31	5.00	4.76	4.57	4.41	4.17	3.92	3.66	3.52	3.38	3.23	3.08	2.92	2.75
28	13.50	8.93	7.19	6.25	5.66	5.24	4.93	4.69	4.50	4.35	4.11	3.86	3.60	3.46	3.32	3.18	3.02	2.86	2.69
29	13.39	8.85	7.12	6.19	5.59	5.18	4.87	4.64	4.45	4.29	4.05	3.80	3.54	3.41	3.27	3.12	2.97	2.81	2.64
30	13.29	8.77	7.05	6.12	5.53	5.12	4.82	4.58	4.39	4.24	4.00	3.75	3.49	3.36	3.22	3.07	2.92	2.76	2.59
40	12.61	8.25	6.60	5.70	5.13	4.73	4.44	4.21	4.02	3.87	3.64	3.40	3.15	3.01	2.87	2.73	2.57	2.41	2.23
60	11.97	7.76	6.17	5.31	4.76	4.37	4.09	3.87	3.69	3.54	3.31	3.08	2.83	2.69	2.55	2.41	2.25	2.08	1.89
120	11.38	7.32	5.79	4.95	4.42	4.04	3.77	3.55	3.38	3.24	3.02	2.78	2.53	2.40	2.26	2.11	1.95	1.76	1.54
∞	10.83	6.91	5.42	4.62	4.10	3.74	3.47	3.27	3.10	2.96	2.74	2.51	2.27	2.13	1.99	1.84	1.66	1.45	1.00