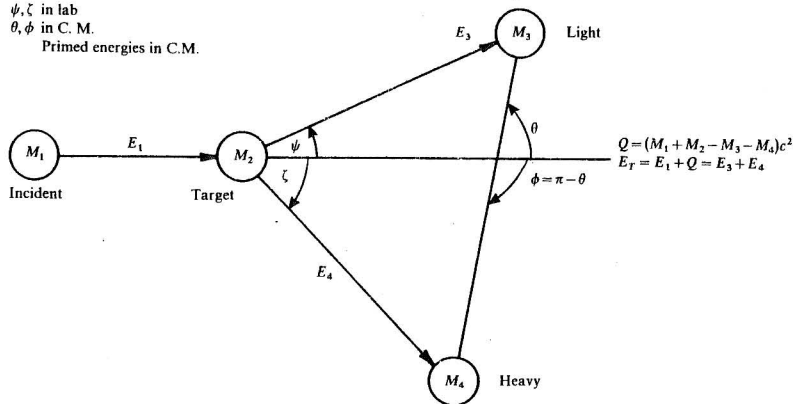


TABLE 5

Kinematics of nuclear reactions and scattering (continued)

 ψ, ζ in lab
 θ, ϕ in C.M.

Primed energies in C.M.



Define:

$$A = \frac{M_1 M_4 (E_1/E_T)}{(M_1 + M_2)(M_3 + M_4)}, \quad C = \frac{M_2 M_3}{(M_1 + M_2)(M_3 + M_4)} \left(1 + \frac{M_1 Q}{M_2 E_T}\right) = \frac{E_4}{E_T}$$

$$B = \frac{M_1 M_3 (E_1/E_T)}{(M_1 + M_2)(M_3 + M_4)}, \quad D = \frac{M_2 M_4}{(M_1 + M_2)(M_3 + M_4)} \left(1 + \frac{M_1 Q}{M_2 E_T}\right) = \frac{E_3}{E_T}$$

Note that $A + B + C + D = 1$ and $AC = BD$

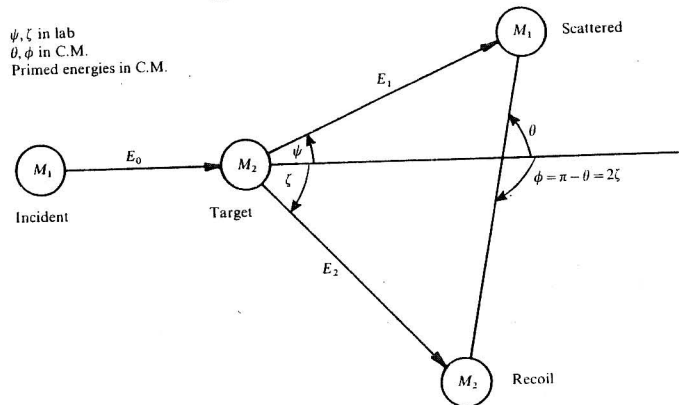
Lab energy of light product:	$\frac{E_3}{E_T} = B + D + 2(AC)^{1/2} \cos \theta$ $= B[\cos \psi \pm (D/B - \sin^2 \psi)^{1/2}]^2$	Use only plus sign unless $B > D$, in which case $\psi_{\max} = \sin^{-1}(D/B)^{1/2}$
Lab energy of heavy product:	$\frac{E_4}{E_T} = A + C + 2(AC)^{1/2} \cos \phi$ $= A[\cos \zeta \pm (C/A - \sin^2 \zeta)^{1/2}]^2$	Use only plus sign unless $A > C$, in which case $\zeta_{\max} = \sin^{-1}(C/A)^{1/2}$
Lab angle of heavy product:	$\sin \zeta = \left(\frac{M_3 E_3}{M_4 E_4}\right)^{1/2} \sin \psi$	C.M. angle of light product: $\sin \theta = \left(\frac{E_3/E_T}{D}\right)^{1/2} \sin \psi$
Intensity or solid-angle ratio for light product:	$\frac{\sigma(\theta)}{\sigma(\psi)} = \frac{I(\theta)}{I(\psi)} = \frac{\sin \psi d\psi}{\sin \theta d\theta} = \frac{\sin^2 \psi}{\sin^2 \theta} \cos(\theta - \psi) = \frac{(AC)^{1/2}(D/B - \sin^2 \psi)^{1/2}}{E_3/E_T}$	
Intensity or solid-angle ratio for heavy product:	$\frac{\sigma(\phi)}{\sigma(\zeta)} = \frac{I(\phi)}{I(\zeta)} = \frac{\sin \zeta d\zeta}{\sin \phi d\phi} = \frac{\sin^2 \zeta}{\sin^2 \phi} \cos(\phi - \zeta) = \frac{(AC)^{1/2}(C/A - \sin^2 \zeta)^{1/2}}{E_4/E_T}$	
Intensity or solid-angle ratio for associated particles in the lab system:	$\frac{\sigma(\zeta)}{\sigma(\psi)} = \frac{I(\zeta)}{I(\psi)} = \frac{\sin \psi d\psi}{\sin \zeta d\zeta} = \frac{\sin^2 \psi \cos(\theta - \psi)}{\sin^2 \zeta \cos(\phi - \zeta)}$	

TABLE 5

Kinematics of nuclear reactions and scattering (continued)

 ψ, ζ in lab
 θ, ϕ in C.M.

Primed energies in C.M.



$$E_1' = \frac{M_1^2}{(M_1 + M_2)^2} E_0$$

For elastic scattering, all energy and angle ratios are independent of energy and reduce as below:

$$E_2' = \frac{M_1 M_2}{(M_1 + M_2)^2} E_0$$

Lab energy of the scattered particle:	$\frac{E_1}{E_0} = 1 - \frac{2M_1 M_2}{(M_1 + M_2)^2} (1 - \cos \theta)$ $= \frac{M_1^2}{(M_1 + M_2)^2} [\cos \psi \pm [(M_2/M_1)^2 - \sin^2 \psi]^{1/2}]^2$	Use only plus sign unless $M_1 > M_2$, in which case $\psi_{\max} = \sin^{-1}(M_2/M_1)$
Lab energy of the recoil nucleus:	$\frac{E_2}{E_0} = 1 - E_1/E_0 = \frac{4M_1 M_2}{(M_1 + M_2)^2} \cos^2 \zeta$	$\zeta \leq \frac{1}{2}\pi$
Lab angle of recoil nucleus:	$\sin \zeta = \left(\frac{M_1 E_1}{M_2 E_2}\right)^{1/2} \sin \psi$	$\zeta = \frac{1}{2}(\pi - \theta)$, $\tan \psi = \frac{\sin 2\zeta}{M_1/M_2 - \cos 2\zeta}$
C.M. angle of scattered particle:	$\theta = \psi + \sin^{-1} \left(\frac{M_1}{M_2} \sin \psi \right) = \pi - 2\zeta$	$\cos \theta = 1 - 2 \cos^2 \zeta$
Intensity or solid-angle ratio for scattered particle:	$\frac{\sigma(\theta)}{\sigma(\psi)} = \frac{I(\theta)}{I(\psi)} = \frac{\sin \psi d\psi}{\sin \theta d\theta} = \frac{\sin^2 \psi}{\sin^2 \theta} \cos(\theta - \psi) = \frac{M_1 M_2 [(M_2/M_1)^2 - \sin^2 \psi]^{1/2}}{(M_1 + M_2)^2 (E_1/E_0)}$	
Intensity or solid-angle ratio for recoil nucleus:	$\frac{\sigma(\phi)}{\sigma(\zeta)} = \frac{I(\phi)}{I(\zeta)} = \frac{\sin \zeta d\zeta}{\sin \phi d\phi} = \frac{1}{4 \cos^2 \zeta}$	

frames of reference, the laboratory system and the center-of-mass system. In heavy-ion collisions, the two partners are of comparable mass; hence an appreciable fraction of the total kinetic energy is tied up in translational motion of the system as a whole, and the transformation from one system to the other has important consequences.

In the following we consider only collisions which result in the emission of two fragments, as in elastic or inelastic scattering, or simple transfer reactions.

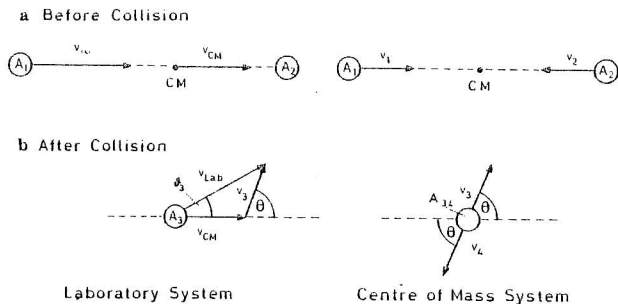


Fig. 1.1. Velocities and angles in the laboratory and centre-of-mass frames of reference

The basic features are illustrated in Fig. 1.1. The two upper diagrams represent the situation before the collision. In the laboratory system (left), the target (mass number A_2) is at rest, while the projectile (mass number A_1) approaches with an initial velocity v_∞ . The kinetic energy and momentum are given by

$$E_{\text{Lab}} = \frac{m}{2} A_1 v_\infty^2, \quad P_{\text{Lab}} = m A_1 v_\infty, \quad (1.1)$$

where m is the nucleon mass and the significance of the other symbols is obvious. In the centre-of-mass system (right) the two ions move with initial velocities

$$v_1 = \frac{A_2}{A_1 + A_2} v_\infty, \quad v_2 = \frac{A_1}{A_1 + A_2} v_\infty, \quad (1.2)$$

and the kinetic energy and relative momentum are

$$E_{\text{CM}} = \frac{m}{2} (A_1 v_1^2 + A_2 v_2^2) = \frac{\mu}{2} v_\infty^2, \quad (1.3)$$

$$P_{\text{CM}} = m (A_1 v_1 + A_2 v_2) = \mu v_\infty. \quad (1.4)$$

The symbol μ denotes the reduced mass which is given by

$$\mu = m \frac{A_1 A_2}{A_1 + A_2} = A_{12} m, \quad (1.5)$$

where A_{12} may be termed the reduced mass number of the colliding system.

Another very useful quantity, which will frequently appear in this book, is the laboratory energy of the projectile divided by its mass number and normally measured in MeV per nucleon:

$$e = \frac{E_{\text{Lab}}}{A_1} = \frac{E_{\text{CM}}}{A_{12}} = \frac{m}{2} v_\infty^2. \quad (1.6)$$

It should be noted that only the centre-of-mass values of energy and momentum are available for interactions between the two ions; they represent a fraction of the corresponding laboratory values equal to the ratio of reduced mass to projectile mass.

The lower left diagram in Fig. 1.1 refers to the asymptotic motion of any fragment with mass number A_3 emerging after the collision. Its final velocity in the laboratory, v_{Lab} , is the vector sum of its final velocity in the centre-of-mass system, v_3 , and the translational velocity of the centre-of-mass system with respect to the laboratory system, v_{CM} . The magnitude of the latter is given by

$$v_{\text{CM}} = v_2 = \frac{A_1}{A_1 + A_2} v_\infty. \quad (1.7)$$

The emission angles of fragment 3 with respect to the incident beam direction, ϑ_3 (in the laboratory system) and θ (in the centre-of-mass system), are related by

$$\tan \vartheta_3 = \frac{v_3 \sin \theta}{v_3 \cos \theta + v_{\text{CM}}} = \frac{\sin \theta}{\cos \theta + \gamma_3}, \quad (1.8)$$

where $\gamma_3 = v_{\text{CM}}/v_3$.

In the special case of elastic scattering (or, more generally, in collisions with negligible mass and energy transfer), the magnitude of the centre-of-mass velocity is not changed ($v_3 = v_1$), and hence we obtain from (1.2), (1.7), and (1.8)

$$\gamma_3 = \frac{v_{\text{CM}}}{v_3} = \frac{A_1}{A_2}. \quad (1.9)$$

If further the two colliding ions are of equal mass, we have $\gamma_3 = 1$, $\vartheta_3 = \theta/2$. In the general case of a two-body reaction with product mass numbers A_3 and A_4 , the quantity γ_3 is given by [Sc 55]

$$\gamma_3 = + \left(\frac{A_1 A_3}{A_2 A_4} \frac{E_{\text{CM}}}{E_{\text{CM}} + Q} \right)^{1/2} \quad (1.10)$$

where Q denotes the reaction Q value.

Next we consider scattering in the Coulomb field. Figure 1.2 shows the

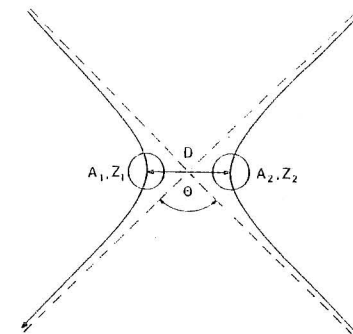


Fig. 1.2. Coulomb trajectories in the centre-of-mass system

classical trajectories of two colliding ions in the centre-of-mass system. The distance of closest approach between the two centres of gravity, D , and the scattering angle θ are related by

$$D = a \left(1 + \csc \frac{\theta}{2} \right), \quad (1.11)$$

where the parameter a is defined as one-half the distance of closest approach in a head-on collision ($\theta = 180^\circ$):

$$a = \frac{Z_1 Z_2 e^2}{\mu v_\infty^2} \quad (1.12)$$

Together with the asymptotic wavelength of relative motion at large separation, $\lambda = h/\mu v_\infty$, we obtain the Sommerfeld parameter

$$n = \frac{a}{\lambda} = \frac{Z_1 Z_2 e^2}{\hbar v_\infty}. \quad (1.13)$$

According to our discussion in the preceding section, large values of n correspond to nearly classical motion.

For a given pair of colliding ions, the Coulomb trajectories are completely specified by the two kinematic parameters D and θ . However, it is often more convenient to use instead the two constants of motion: the total energy

$E_{CM} = \frac{1}{2} \mu v_a^2$, and the (classical) angular momentum $L\hbar$ [here and in the following, we use L for classical and l for quantized angular momenta in units of $\hbar$; they are related by $L^2 = l(l+1) \approx (l + \frac{1}{2})^2$]. The requirement of energy conservation from infinity to the point of closest approach leads to the following relationship:

$$E_{CM} = \frac{Z_1 Z_2 e^2}{D} + \frac{L^2 \hbar^2}{2\mu D^2}, \quad (1.14)$$

which can be rearranged, with the help of the previously introduced Sommerfeld parameter n (1.13), to yield

$$L^2 = kD(kD - 2n), \quad (1.15a)$$

$$kD = n + (n^2 + L^2)^{1/2}. \quad (1.15b)$$

Equations (1.15) establish an important relationship between the distance of closest approach and angular momentum for Coulomb trajectories, which is often used in the semiclassical analysis of heavy ion scattering.

Likewise we can express the scattering angle Θ in terms of n and L ; from (1.11), (1.13) and (1.15 b), we obtain

$$\sin \frac{\Theta}{2} = \frac{n}{kD - n} = \frac{n}{(n^2 + L^2)^{1/2}}. \quad (1.16)$$

At some internuclear distance $D \approx R_C$, the nuclear interaction will become effective. We define the Coulomb barrier E_C as the asymptotic kinetic energy in the centre-of-mass system at the (classical) threshold for nuclear reactions, and the Coulomb interaction distance $R_C = r_{0C} (A_1^{1/3} + A_2^{1/3})$ as the corresponding distance of closest approach in the absence of nuclear interactions. From (1.3) and (1.12) we obtain, with $2a = R_C$ and $E_{CM} = E_C$,

$$E_C = A_{12} e_C = \frac{Z_1 Z_2 e^2}{R_C}, \quad (1.17)$$

which is, of course, just the Coulomb potential between two spherically symmetric, nonpenetrating charge distributions with total charges $Z_1 e$ and $Z_2 e$ at a distance R_C . Due to the diffuse nature of the nuclear surface and the finite range of the nuclear interaction, R_C will in general be significantly larger than the sum of the effective matter radii of the colliding nuclei; moreover, R_C will be larger than the actual distance of closest approach (R_{in}) at $\varepsilon = \varepsilon_C$, since we have neglected deflections by the attractive nuclear field in its definition. A quantitative discussion of the distances R_C and R_{in} will follow in Sect. 3.3.

At bombarding energies above the Coulomb barrier, we have to distinguish two different kinematic situations, depending on whether the distance of closest approach—assuming pure Coulomb scattering of point charges—is

larger or smaller than R_C . For large impact parameters, corresponding to small scattering angles, we have Coulomb trajectories as shown in Fig 1.2. For small impact parameters, on the other hand, the collision is dominated by nuclear interactions and, in general, leads to inelastic processes. The limiting case of a "grazing collision" ($D = R_C$) is of particular interest; from (1.16) we obtain for the corresponding Coulomb scattering angle Θ_{gr} ,

$$\sin \frac{\Theta_{gr}}{2} = \frac{n}{kR_C - n} = \frac{e_C}{2\varepsilon - e_C}. \quad (1.18)$$

In (1.18) we have made use of the identity

$$kR_C = 2n \frac{\varepsilon}{e_C} = 2n_C \left(\frac{\varepsilon}{e_C} \right)^{\frac{1}{2}}, \quad (1.19)$$

where n_C is the Sommerfeld parameter taken at the Coulomb barrier. It is often convenient to describe situations above the Coulomb barrier in terms of the parameters kR_C and ε/e_C , rather than n , which is more appropriate for pure Coulomb scattering. A good example is (1.18), which shows that the grazing angle depends only on the ratio ε/e_C .

The classical angular momentum L_{gr} , corresponding to the grazing trajectory can be derived from (1.15a) and (1.19) as

$$L_{gr}^2 = (kR_C)^2 \left(1 - \frac{e_C}{\varepsilon} \right) = 4n_C^2 \left(\frac{\varepsilon}{e_C} - 1 \right) \quad (1.20)$$

We conclude that for any given pair of ions the grazing angular momentum vanishes at the Coulomb barrier ($\varepsilon = e_C$), where grazing implies a head-on collision, and rises at higher energies proportional to $(\varepsilon - e_C)^{1/2}$. For given values of e_C and $\varepsilon > e_C$, the angular momentum L_{gr} is proportional to the product $A_{12} R_C$.

From simple geometrical arguments we can derive the momentum transfer in an elastic grazing collision q_{gr} as

$$q_{gr} = 2k \sin \frac{\Theta_{gr}}{2} = 2k \frac{e_C}{2\varepsilon - e_C}. \quad (1.21)$$

In contrast to the angular momentum, this quantity decreases with increasing bombarding energy above the Coulomb barrier, as a consequence of the decrease in the scattering angle Θ_{gr} .

We conclude this discussion by noting that we have used the concept of a Coulomb interaction distance somewhat loosely, by calculating the Coulomb barrier (1.17) and the grazing trajectory (1.18) with the same quantity R_C . Since we are not interested in detailed predictions at this point, such a simplified description is considered adequate.

In order to give an impression of the magnitudes involved, we have collected in Table 1.1 numerical values of various quantities discussed in this section