Dynamical Casimir effect in 2011: main achievements and challenges in theory and experiment

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What is the “dynamical Casimir effect”? 

Phenomena caused by changes of \textit{(vacuum)} quantum states of fields 
due to \textbf{fast} time variations of positions 
(or properties) of boundaries confining the fields 
(or other parameters) 

Modification of the Casimir force for moving boundaries 

Creations of the field quanta (photons) 
due to the motion of boundaries
Why it is interesting?

• A direct proof of the existence of vacuum fluctuations?
  (Indirect available proofs: Lamb shift, stationary Casimir force, etc.)

• This is a challenge for theoreticians and experimentalists

• It stimulates studies of many problems in different fields of quantum and classical physics
The beginning


• The first attempts to think about experiments

- Nonstationary Casimir effect


The name **Dynamical Casimir effect** was invented by Yablonovitch and Schwinger:

‘... we are considering sudden nonadiabatic changes which have the effect of causing real transitions and boosting the quantum fluctuations into real photons. In that sense this process may be called the *dynamic or nonadiabatic Casimir effect*.’

The impetus to fill that gap comes from recent discoveries in coherent sonoluminescence (5) that I interpret as a *dynamical Casimir effect* wherein dielectric media are accelerated and emit light.

-Mirror induced radiation:

-Motion induced radiation:

-Motional Casimir force:
1990s: reasonable calculations of the photon production rate in idealized situations. The idea of parametric amplification


Effective Hamiltonian approach


If Maxwell’s equation in a medium with time-independent parameters and boundaries can be reduced to

\[ \hat{\mathcal{K}}(\{L\}) \mathbf{F}_\alpha(\mathbf{r}; \{L\}) = \omega^2_\alpha(\{L\}) \mathbf{F}_\alpha(\mathbf{r}; \{L\}) \]

Then for \{L(t)\} and the Dirichlet boundary conditions:

\[ \mathbf{F}(\mathbf{r}, t) = \sum_\alpha q_\alpha(t) \mathbf{F}_\alpha(\mathbf{r}; \{L(t)\}) \]

\[ H = \frac{1}{2} \sum_\alpha \left[ p^2_\alpha + \omega^2_\alpha(L(t)) q^2_\alpha \right] + \frac{\dot{L}(t)}{L(t)} \sum_{\alpha \neq \beta} p_\alpha m_{\alpha \beta} q_\beta, \]

\[ m_{\alpha \beta} = -m_{\beta \alpha} = L \int dV \frac{\partial \mathbf{F}_\alpha(\mathbf{r}; L)}{\partial L} \mathbf{F}_\beta(\mathbf{r}; L). \]
Two coupled modes

Resonant photon creation in a three-dimensional oscillating cavity

Martin Crocce,1,* Diego A. R. Dalvit,2,† and Francisco D. Mazzitelli1,‡

When the mode \( k \) is coupled to one mode, the rate of photon creation decreases by a factor of 2 with respect to the uncoupled case.

Nonstationary Casimir effect in cavities with two resonantly coupled modes

A.V. Dodonov, V.V. Dodonov*1

Dynamical Casimir effect for TE and TM modes in a resonant cavity bisected by a plasma sheet

W. Naylor,1,* S. Matsuki,1 T. Nishimura,2 and Y. Kido1
Fabry - Perot cavity with harmonically oscillating boundaries

\[ x_{left} \equiv 0, \quad x_{right} \equiv L(t) = L_0 \left( 1 + \varepsilon \sin [p \omega_1 t] \right) \]

\[ |\varepsilon| \ll 1; \quad p = 1, 2, \ldots (\omega_1 = \pi c / L_0); \quad \omega_k = k \omega_1 \]

TE mode - the Dirichlet boundary conditions

\[ \hat{A}(x, t) = \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} \left[ \hat{b}_n \psi^{(n)}(x, t) + \text{h.c.} \right] \]

\[ \psi^{(n)}(x, t) = \sqrt{\frac{L_0}{L(t)}} \sum_{k=1}^{\infty} \sin \left[ \frac{\pi k x}{L(t)} \right] \left\{ \rho^{(n)}_k e^{-i \omega_k t} - \rho^{(n)}_{-k} e^{i \omega_k t} \right\} \]

\[ \frac{d \rho^{(n)}_k}{d \tau} = (-1)^p \left[ (k + p) \rho^{(n)}_{k+p} - (k - p) \rho^{(n)}_{k-p} \right], \quad \tau = \frac{1}{2} \varepsilon \omega_1 t \]

\[ \rho^{(j+np)}_{j+m\sigma} (\tau) = \frac{\Gamma \left( 1 + n + \frac{j}{p} \right) \left( \sigma \kappa \right)^{n-m}}{\Gamma \left( 1 + m + \frac{j}{p} \right) \Gamma \left( 1 + n - m \right)} \times F \left( n + \frac{j}{p}, -m - \frac{j}{p}; 1 + n - m; \kappa^2 \right) \]

\[ \sigma \equiv (-1)^p \]

\[ \kappa = \tanh (p \tau) \]

\[ \mathcal{E}^{(\text{vac})}(\tau) = \frac{p^2 - 1}{12a^2} \sinh^2(pa\tau) \]

\[ \mathcal{N}_1^{(\text{vac})}(\tau \gg 1) = \frac{8a}{\pi^2} \tau + \frac{4}{\pi^2} \ln\left(\frac{2}{a}\right) - \frac{1}{2} + O(\tau e^{-4a\tau}) \]

Physical realizations: TEM modes in a coaxial cavity

Crocce M, Dalvit D A R, Lombardo F C and Mazzitelli F D
Small squeezing in each mode:

\[
U_1(\infty) = \frac{2}{\pi^2} \approx 0.20;
\]

\[
U_3(\infty) = \frac{38}{(9\pi^2)} \approx 0.43;
\]

\[
\frac{1}{2} - U_{2m+1}(\infty) \sim \frac{1}{(2m + 1)}
\]
Formation of sharp packets

V.N. Krasil'nikov, 1968;

A.I. Vesnitskii, 1971;

C.K. Law, PRL 73, 1931 (1994);


A. Lambrecht, M.-T. Jaekel and S. Reynaud, Europhys. Lett. 43, 147 (1998);

Figure 2. The energy density for the initial vacuum state in the case of the strict resonance $\gamma = 0$, for $[t] = 0.3$ and $\kappa = 0.9$; $p = 2$ (full curve) and $p = 3$ (broken curve).
For harmonical resonance oscillations, $\omega_\omega = 2\omega_0$:

$$\langle n \rangle (t) = \sinh^2 (\varepsilon \omega_0 t \eta^3), \quad \eta = \lambda / (2L_0) < 1$$

$\varepsilon = a / \lambda$ - the maximal relative displacement of the boundary


For oscillations of the surface of the cavity wall:

$$\delta = \omega_0 a / v_s \leq 10^{-2}, \quad v_{\text{max}} \sim \delta_{\text{max}} v_s \sim 50 \text{ m/s}$$

$$\varepsilon_{\text{max}} \sim \left( v_s / 2\pi c \right) \delta_{\text{max}} \sim 3 \times 10^{-8}$$

If $\varepsilon = 10^{-9}$, $\omega_0 / (2\pi) = 10 \text{ GHz}$, $\eta \sim 1/2$, $t = 1 \text{ s}$, $Q \sim 10^{10}$, $\delta \omega / (2\pi) < \varepsilon \omega_0 / (2\pi) \sim 3 \text{ Hz}$

$$\langle n \rangle \sim \sinh^2 (10) \sim 10^8$$
Detectability of Dissipative Motion in Quantum Vacuum via Superradiance

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We propose an experiment for generating and detecting vacuum-induced dissipative motion. A high frequency mechanical resonator driven in resonance is expected to dissipate mechanical energy in quantum vacuum via photon emission. The photons are stored in a high quality electromagnetic cavity and detected through their interaction with ultracold alkali-metal atoms prepared in an inverted population of hyperfine states. Superradiant amplification of the generated photons results in a detectable radio-frequency signal temporally distinguishable from the expected background.

(a) Inversion of atomic population  (b) Generation and amplification of Casimir photons  (c) Detection of superradiant photons or ground state population
Bulk acoustic resonator with thin AlN film

Bunches of $10^8$ Rydberg atoms ($n \sim 50$) as detectors

$\varepsilon \sim 10^{-8}$  $\nu \sim 3$ GHz
Effective electron-hole “plasma mirror”


Padua experiment (MIR) since 2002


Prospects of employing superconducting stripline resonators for studying the dynamical Casimir effect experimentally

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Abstract

We discuss the prospects of employing an NbN superconducting microwave stripline resonator for studying the dynamical Casimir effect experimentally. Preliminary experiments, in which optical illumination is employed for modulating the resonance frequencies of the resonator, show that such a system is highly promising for this purpose. In addition, we discuss the undesirable effect of heating which results from the optical illumination, and show that degradation in noise properties can be minimized by employing an appropriate design.
A concrete proposal relating DCE with the laser-illuminated superconductors was made in [93], where the superconducting stripline resonator (a ring having a radius of 6.39 mm and width 347 \( \mu \)m composed of an NbN film of 8 nm thickness deposited on a sapphire wafer [94]) was considered as a promising candidate for the photon generation from vacuum in the range from 2 to 8 GHz. Strictly speaking, it is difficult to connect the change of the frequency shift of this resonator with an effective motion of some boundary. But if one assumes the definition of the DCE as the phenomenon of photon creation from vacuum due to the change of some parameters of a system [14], then this scheme fits perfectly to the PA-DCE family. The advantages of proposals based on the periodic illumination of superconductors consist of the easy modulation of the resonance frequency, the big amplitude of its variations and a low necessary energy of laser pulses. For example, a parabolic dependence of frequency shift on pulse energy was reported in [95]. The 70 ps pulses of energy 3 nJ resulted in a 20 MHz shift at the temperature of 20 K and almost 100 MHz at 80 K (for the YBa\(_2\)Cu\(_3\)O\(_{7-x}\) strips). The NbN films demonstrated [93] an almost 40 MHz frequency shift (at the liquid helium temperature), caused by pulses of the infrared laser (1550 nm wavelength) modulated at twice the resonator eigenfrequency 7.74 GHz. The reported laser power was 27 nW.
Analogue Casimir radiation using an optical parametric oscillator

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PACS 12.20.−m – Quantum electrodynamics

Abstract – We establish an explicit analogy between the dynamical Casimir effect and the photon emission of a thin non-linear crystal pumped inside a cavity. This allows us to propose a system based on a type-I optical parametric oscillator (OPO) to simulate a cavity oscillating in vacuum at optical frequencies. The resulting photon flux is expected to be more easily detectable than with a mechanical excitation of the mirrors. We conclude by comparing different theoretical predictions and suggest that our experimental proposal could help discriminate between them.
The photon flux radiated outside the one-dimensional Fabry–Pérot cavity with harmonically oscillating semitransparent boundaries was calculated in [9]. Using the spectral approach, it was shown that the radiation can be essentially enhanced under the resonance conditions, comparing with the case of a single oscillating mirror. It was suggested recently [112] that the motion of the boundary could be imitated by putting inside the cavity a thin nonlinear crystal (thickness $0.1 \mu m$ and nonlinear susceptibility $\chi^{(2)} \sim 10^{-11} m/V$), pumped by an optical beam of frequency $f = \Omega/2\pi \approx 3 \times 10^{14}$ Hz ($\lambda = c/f = 1 \mu m$) and power about 1 W, focalized over an area $A = 10^{-10} m^2$ (the ‘Casimir’ photons can be distinguished from the pump ones due to the orthogonal polarizations). The evaluations were made in the framework of the one-dimensional model of the cavity (because the equidistant spectrum of eigenfrequencies was used explicitly). Consequently, the cavity must be very small: its length $L$ must obey the inequality $L \ll \sqrt{A} = 10 \mu m$. Actually, this inequality together with the resonance conditions can be satisfied in the case involved only for $L = \lambda$. These values of parameters result in the amplitude of variation of the effective cavity length $\Delta L \sim 10^{-12} m$ and the relative maximum velocity of the equivalent moving boundary $\beta = v/c \sim 10^{-6}$. Then the formulae derived in [9] give the following average total flux of ‘Casimir’ photons leaving the cavity with finesse $F = 10^4$ after time $t$: $\langle N_{\text{out}} \rangle/t = \beta^2 F \Omega/(3\pi) \sim 10^5$ photons per second (accidentally the same number as for the microwave photons in the scheme of [102]). As a matter of fact this is not a big number, because it means that photons are emitted with intervals of about $10^{-5}$ s, so it is necessary to wait for $100 \mu s$ in order to register about ten photons. This is explained by the low stationary mean number of photons inside the cavity: according to [9] $\langle N_{\text{in}} \rangle = 2(\beta F)^2/(3\pi^2) \sim 10^{-5}$, and this evaluation follows also from formula (7), if one identifies $t$ with the relaxation time of the leaking cavity $t \sim F/\omega_1$ and puts $\kappa = \Delta L/(2L) = 5 \times 10^{-7}$. 
For comparison, in the MIR experiment (discussed in subsection 3.2) it is expected to generate from $10^3$ to $10^4$ microwave quanta after 1000–2000 laser pulses of total duration 0.2–0.4 $\mu$s and total energy about 10–20 mJ. To emit the same amount of photons from the Fabry–Pérot cavity under consideration, one needs from 10 to 100 mJ in the pumping laser beam.
Dynamical Casimir Effect in a Superconducting Coplanar Waveguide

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Dynamical Casimir effect in superconducting microwave circuits

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Photon Generation in an Electromagnetic Cavity with a Time-Dependent Boundary

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Observation of the Dynamical Casimir Effect in a Superconducting Circuit

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(Dated: May 25, 2011)
Many effects of the circuit QED can be understood in the framework of a simple model with the interaction Hamiltonian 
\[ \hat{H}_{\text{int}} = \hbar g(t)(\hat{a} + \hat{a}^\dagger)(\hat{\sigma}_+ + \hat{\sigma}_-) \] [104] (where \( \hat{\sigma}_+ \) and \( \hat{\sigma}_- \) are the raising and lowering operators describing atomic (electron) excitations) or its generalizations [96]. Since this Hamiltonian is formally \textit{linear} with respect to the photon creation and annihilation operators \( \hat{a}^\dagger \) and \( \hat{a} \), one can expect, in principle, a higher photon generation rate than in the case of the frequency variation (where the interaction Hamiltonian is \textit{quadratic} with respect to \( \hat{a}^\dagger \) and \( \hat{a} \)), provided a strong modulation of the effective Rabi frequency \( g(t) \) can be achieved. It seems that such a possibility exists: see the next subsection.
Quantum vacuum properties of the intersubband cavity polariton field

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Quantum Vacuum Radiation Spectra from a Semiconductor Microcavity with a Time-Modulated Vacuum Rabi Frequency

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Dynamical Casimir effect for TE and TM modes in a resonant cavity bisected by a plasma sheet

W. Naylor, S. Matsuki, T. Nishimura, and Y. Kido

Photon creation in a resonant cavity with a nonstationary plasma mirror and its detection with Rydberg atoms

Toru Kawakubo and Katsuji Yamamoto
Padua experiment (MIR): Theoretical model and problems
Semiconductor time-dependent mirror

\[ \varepsilon(x) = \varepsilon_1 + i \varepsilon_2, \quad \varepsilon_2 = \frac{2\sigma}{f}, \]

- \(\sigma\) - conductivity in the CGS units
- \(f\) - frequency in Hz

at 2.3 GHz, \(\varepsilon_2 \sim 10^5-10^6\) \(\varepsilon_1 \sim 10\)

Laser wavelength \(\sim 0.8 \mu m\), “Casimir wavelength” \(\lambda \sim 10\) cm

\[ \varepsilon_1 = \text{const}, \quad \sigma(x, t) = n(x, t)eb \]

\[ \varepsilon(\omega) = \varepsilon_a + \frac{4\pi i \sigma_0}{\omega(1 - i\omega\tau)}, \quad \sigma_0 = ne^2\tau/m, \]

\[ \Omega(t) = \omega(t) - i\gamma(t) \]
How to calculate \[ \Omega(t) = \omega(t) - i\gamma(t) \]
\[ \Omega(t) = \omega(t) - i \gamma(t) \]

\[ \chi(t) \approx \frac{\zeta_m A^2(t)}{A^2(t) + 1}, \]

\[ \gamma_s(t) \approx \frac{\omega_0 |\zeta_m| A(t)}{A^2(t) + 1} \]

\[ A(t) = A_0 \exp\left(-\frac{t}{T_r}\right) f(t) \]
Figure 1. The relative shift of the cavity eigenfrequency $\chi$ (upper curves) and normalized damping coefficient $\gamma$ (lower curves) versus the dimensionless time variable $\zeta = \beta_1 t = \tau/Z$ for $A_0 = 10$ and fixed values of the parameters $g$ and $h$ (from right to left): $g = 0$ and $h$ arbitrary; $g = 10$ and $h = 1$; $g = 10$ and $h = 10$; $g = 10$ and $h = 100$. 
Padua experiment (MIR) -2010:

- Reentrant cavity
- $f \sim 2.33 \text{ GHz}$
\[ \omega(t) \equiv \omega_0[1 + \chi(t)] \quad \Omega(t) = \omega(t) - i\gamma(t) \]

\[ \chi(t) \approx \frac{\zeta_m A^2(t)}{A^2(t) + 1}, \]

\[ A(t) = A_0 \exp(-t/T_r)f(t) \]

\[ \gamma_s(t) \approx \frac{\omega_0 |\zeta_m| A(t)}{A^2(t) + 1} \]

\[ A_0 = \varepsilon_b Y K(D/\lambda) \]

\[ K = \left(\frac{\lambda}{D}\right)^2 \frac{\int \int_S |E_{t0}(x,y,0)|^2 dS}{\int \int_S |E_{z0}(x,y,0)|^2 dS} \]

\[ \zeta_m \approx -\frac{D \int \int_S |E_{z0}(x,y,0)|^2 dS}{2\varepsilon_b \int \int \int_{cav} |E_0(r)|^2 dV} \approx \zeta_{id}/\varepsilon_b \]

\[ Y = \frac{2|eb|\kappa W}{cE_g S} \]
Nonstationary quantum **damped** oscillator

\[ \Omega(t) = \omega(t) - i\gamma(t) \quad \Omega(0) = 1 \]

\[
\begin{align*}
\frac{d\hat{x}}{dt} &= \hat{p} - \gamma_x(t)\hat{x} + \hat{F}_x(t), \\
\frac{d\hat{p}}{dt} &= -\gamma_p(t)\hat{p} - \omega^2(t)\hat{x} + \hat{F}_p(t)
\end{align*}
\]

\[
\langle \hat{F}_j(t)\hat{F}_k(t') \rangle = \delta(t - t')\chi_{jk}(t), \quad j, k = x, p,
\]

\[ 
\frac{d\hat{a}}{dt} = (-i\omega - \gamma)\hat{a} + \hat{F}_a.
\]
Strictly speaking, the influence of the environment can be reduced to simple local terms only approximately. In general, the evolution is non-Markovian, resulting in the damping terms of the form

\[ \int_{-\infty}^{t} \tilde{\gamma}(t; s) \hat{p}(s) \, ds \]

However, in the specific conditions of the MIR experiment the nonlocal effects seem to be not very important. The reason is the short duration of the dissipative effects: it is determined by the duration of laser pulses and the recombination time of carriers created inside the semiconductor. Both these parameters should be much smaller than the period of free oscillations of the field mode. But the main contribution to the integral is given by the interval

\[ t - s \sim T_r, \text{ since } \tilde{\gamma}(t; s) \approx 0 \text{ if } t - s \gg T_r \]

In addition, the variations of the frequency are very small. This means that the function \( \hat{p}(s) \) practically does not change in the intervals of the order of recombination time. Therefore one can write

\[ \int_{-\infty}^{t} \tilde{\gamma}(t; s) \hat{p}(s) \, ds \approx \hat{p}(t) \int_{-\infty}^{t} \tilde{\gamma}(t; s) \, ds = \hat{p}(t) \gamma(t) \]
\[ \hat{x}(t) = e^{-\Gamma(t)} \left\{ \hat{x}_0 \text{Re} [\xi(t)] - \hat{p}_0 \text{Im} [\xi(t)] \right\} + \hat{X}(t) \]

\[ \hat{p}(t) = e^{-\Gamma(t)} \left\{ \hat{x}_0 \text{Re} [\eta(t)] - \hat{p}_0 \text{Im} [\eta(t)] \right\} + \hat{P}(t) \]

\[ \Gamma(t) = \int_{-\infty}^{t} \gamma(\tau) d\tau, \]

\[ \gamma(t) = \frac{1}{2} \left[ \gamma_x(t) + \gamma_p(t) \right] \]

\( \xi(t) \) satisfies equation

\[ \ddot{x} + \omega^2(t)x = 0. \]

\( \dot{\xi} \dot{\xi}^* - \dot{\xi}^* \dot{\xi} = 2i \)

\[ \omega_{ef}^2(t) = \omega^2(t) + \delta(t) - \delta^2(t), \quad \delta(t) = \frac{1}{2} \left[ \gamma_x(t) - \gamma_p(t) \right] \]
\[\hat{X}(t) = e^{-\Gamma(t)} \int_{-\infty}^{t} d\tau e^{\Gamma(\tau)} \Im \left\{ \xi^*(t) \left[ \hat{F}_p(\tau)\xi(\tau) - \hat{F}_x(\tau)\eta(\tau) \right] \right\}\]

\[\hat{P}(t) = e^{-\Gamma(t)} \int_{-\infty}^{t} d\tau e^{\Gamma(\tau)} \Im \left\{ \eta^*(t) \left[ \hat{F}_p(\tau)\xi(\tau) - \hat{F}_x(\tau)\eta(\tau) \right] \right\}\]

\[[\hat{x}(t), \hat{p}(t)] = i\hbar e^{-2\Gamma(t)} + [\hat{X}(t), \hat{P}(t)]\]

\[\langle [\hat{X}(t), \hat{P}(t)] \rangle = e^{-2\Gamma(t)} \int_{-\infty}^{t} [\chi_{xp}(\tau) - \chi_{px}(\tau)] e^{2\Gamma(\tau)} d\tau\]

\[\chi_{xp}(t) - \chi_{px}(t) = 2i\hbar \dot{\Gamma}(t) \equiv 2i\hbar \gamma(t),\]

\[\langle [\hat{X}(t), \hat{P}(t)] \rangle = i\hbar \left[ 1 - e^{-2\Gamma(t)} \right]\]
JUSTIFICATION OF THE “SYMMETRIC DAMPING” MODEL OF THE DYNAMICAL CASIMIR EFFECT IN A CAVITY WITH A SEMICONDUCTOR MIRROR*

Oscillator Coupled to Nonstationary Bosonic Bath

\[ \hat{H} = \frac{1}{2} \left[ \hat{p}_0^2 + \omega^2(t)\hat{x}_0^2 \right] + \frac{1}{2} \sum_{i=1}^{N} (\hat{p}_i^2 + \omega_i^2 \hat{x}_i^2) + \sum_{i=1}^{N} (z_i \hat{p}_i \hat{p}_0 + v_i \hat{p}_i \hat{x}_0 + u_i \hat{x}_i \hat{p}_0 + g_i \hat{x}_i \hat{x}_0) \]

where the coupling coefficients \( z_i, v_i, u_i, \) and \( g_i \) can be arbitrary functions of time.
The simplest description of the evolution of quantum systems with multidimensional quadratic Hamiltonians can be achieved in the **Wigner representation**

\[ W(q, t) = \int G(q, q', t) W(q', 0) \, dq', \]

\[ G(q, q', t) = \delta(q - q_*(t; q')) \]

\[ q_*(t; q') = R(t)q' \]

\[ q_*(0; q') = q' \]

\[ \dot{R} = A R, \quad R(0) = I_{2N+2} \]

\[ q = (Q, \xi) \]

\[ R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \]
\[ W(q, 0) = W_0(Q)W_1(\xi) \]

\[ W_1(\xi) = (\det F')^{-1/2} \exp \left[ -\frac{1}{2} \xi F^{-1} \xi \right] \]

\[ W(Q, t) = \int W(Q, \xi, t)d^{2N}\xi/(2\pi)^N \]

averaged propagator

\[ G(Q, Q', t) = (2\pi)^{-n}[\det M_*(t)]^{-1/2} \exp \left[ -\frac{1}{2}(Q - R_{11}Q')[M_*^{-1}(Q - R_{11}Q')] \right] \]

\[ M_*(t) = R_{12}(t)F\tilde{R}_{12}(t) \]
\[ \frac{\partial W}{\partial t} = -\frac{\partial}{\partial Q_\alpha} [(AQ)_\alpha W] + D_{\alpha \beta} \frac{\partial^2 W}{\partial Q_\alpha \partial Q_\beta} \]

\[ A = \dot{R}_{11} R^{-1}_{11} = A_{11} + A_{12} R_{21} R^{-1}_{11} \]

\[ 2D = A_{12} (R_{22} - R_{21} R^{-1}_{11} R_{12}) F \tilde{R}_{12} + R_{12} F \left( \tilde{R}_{22} - \tilde{R}_{21} \tilde{R}^{-1}_{11} \tilde{R}_{12} \right) \tilde{A}_{12} \]

\[ \frac{d\hat{Q}}{dt} = A\hat{Q} + \hat{\chi}(t) \]

\[ \langle \hat{\chi}_\alpha(t) \rangle = 0, \quad \langle \hat{\chi}_\alpha(t) \hat{\chi}_\beta(t') \rangle = \delta(t - t') X_{\alpha\beta} \]

\[ 4D = X + \tilde{X}, \quad X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \]
\[
\mathbf{R}_{22}^{(0)}(t) = \exp (\mathbf{A}_{22}t) = \begin{pmatrix}
\text{diag}(\cos \omega_i t) & \text{diag}(-\omega_i \sin \omega_i t) \\
\text{diag}(\omega_i^{-1} \sin \omega_i t) & \text{diag}(\cos \omega_i t)
\end{pmatrix}
\]

\[
\mathbf{R}_{21}^{(1)}(t) = \exp (\mathbf{A}_{22}t) \int_0^t \exp (-\mathbf{A}_{22}\tau) \mathbf{A}_{21}(\tau) \mathbf{R}_{11}^{(0)}(\tau) d\tau
\]

\[
2D = \mathbf{A}_{12} \mathbf{R}_{22}^{(0)} \mathbf{F} \tilde{\mathbf{R}}_{12}^{(1)} + \mathbf{R}_{12}^{(1)} \mathbf{F} \tilde{\mathbf{R}}_{22}^{(0)} \tilde{\mathbf{A}}_{12}
\]

**Special Case of Very Short Interaction Time**

\[
\mathbf{A}^{(1)}(t) = \mathbf{A}_{11}(t) + \mathbf{A}_{12}(t) \mathbf{R}_{21}^{(1)}(t) = \mathbf{A}_{11}(t) + \mathbf{A}_{12}(t) \int_0^t \mathbf{A}_{21}(\tau) d\tau.
\]

\[
2D = \mathbf{A}_{12}(t) \mathbf{F} \tilde{\mathbf{R}}_{12}^{(1)}(t) + \mathbf{R}_{12}^{(1)}(t) \mathbf{F} \tilde{\mathbf{A}}_{12}(t) = \mathbf{A}_{12}(t) \mathbf{F} \int_0^t \tilde{\mathbf{A}}_{12}(\tau) d\tau + \int_0^t \mathbf{A}_{12}(\tau) d\tau \mathbf{F} \tilde{\mathbf{A}}_{12}(t)
\]
\[ \mu(t) \equiv A^{(1)}(t) - A_{11}(t) = A_{12}(t) \int_0^t A_{21}(\tau) \, d\tau \]

\[ \mu_{11}(t) = \sum_{k=1}^N \int_0^t d\tau \left[ v_k(t) u_k(\tau) - g_k(t) z_k(\tau) \right], \]

\[ \mu_{21}(t) = \sum_{k=1}^N \int_0^t d\tau \left[ u_k(t) z_k(\tau) - z_k(t) u_k(\tau) \right], \]

\[ \mu_{12}(t) = \sum_{k=1}^N \int_0^t d\tau \left[ v_k(t) g_k(\tau) - g_k(t) v_k(\tau) \right], \]

\[ \mu_{22}(t) = \sum_{k=1}^N \int_0^t d\tau \left[ u_k(t) v_k(\tau) - z_k(t) g_k(\tau) \right]. \]

\[ u_k(t) = \nu(t) U_k, \quad v_k(t) = \nu(t) V_k, \quad g_k(t) = \nu(t) G_k, \quad z_k(t) = \nu(t) Z_k, \]

\[ \mu_{12} = \mu_{21} \equiv 0, \quad \mu_{11} \equiv \mu_{22} = \lambda(t) \sum_{k=1}^N (U_k V_k - G_k Z_k), \quad \lambda(t) = \nu(t) \int_0^t \nu(\tau) \, d\tau. \]
These coefficients become especially simple if the interaction Hamiltonian has the RWA form:

\[
\hat{H}_{\text{int}} = \nu(t) \sum_{k=1}^{N} \left( \rho_k \hat{a}_0 \hat{a}_k^\dagger + \rho_k^* \hat{a}_k \hat{a}_0^\dagger \right), \quad \hat{a}_0 = \frac{\omega_0 \hat{x}_0 + i \hat{p}_0}{\sqrt{2 \omega_0}}, \quad \hat{a}_k = \frac{\omega_k \hat{x}_k + i \hat{p}_k}{\sqrt{2 \omega_k}}
\]

\[
D_{11}(t) = \lambda(t) \sum_{k=1}^{N} f_k \left( \omega_k^2 V_k^2 + G_k^2 \right), \quad D_{22}(t) = \lambda(t) \sum_{k=1}^{N} f_k \left( \omega_k^2 Z_k^2 + U_k^2 \right), \quad D_{12}(t) = -\lambda(t) \sum_{k=1}^{N} f_k \left( \omega_k^2 V_k Z_k + G_k U_k \right)
\]

\[
D_{11} = \omega_0^2 D_{22} = \lambda(t) \omega_0 \sum_{k=1}^{N} f_k \omega_k |\rho_k|^2 \quad D_{12} \equiv 0
\]

\[
\gamma_x = \gamma_p = \gamma(t)
\]

\[
\chi_{pp} = \omega_0^2 \chi_{xx} = \gamma(t) \omega_0 G
\]

\[
G = \coth \left[ \frac{\hbar \omega_0}{2 k_B T} \right]
\]
Periodical variations of frequency

\[ \omega^2(t) = \omega^2(t+T) : \text{equidistant } \text{``barriers''} \text{ with period } T \text{ and amplitude reflection and transmission coefficients } r \text{ and } f^{-1} \]

\[ \ddot{x} + \omega^2(t)x = 0. \]

\[ x^{(k)}(t) = a_k e^{i\omega t} + b_k e^{-i\omega t} \]

\[ \begin{pmatrix} a_k \\ b_k \end{pmatrix} = M_k \begin{pmatrix} a_{k-1} \\ b_{k-1} \end{pmatrix} \]

Fig. 1. A typical time dependence of the effective frequency.

\[ \cosh \nu \equiv \pm \text{Re} \left( f e^{i\theta} \right), \quad f \equiv |f| e^{i\varphi}, \quad \theta \equiv \omega_0 T. \]
\[ \mathcal{N} = \frac{1}{2} \langle \hat{p}^2 + \hat{x}^2 - 1 \rangle \]

\[ \sigma_N = \langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2, \]

\[ \mathcal{N}^{(th)}(n) = \frac{1}{4} e^{2\nu(n - \Lambda)} \left( G_0 + \frac{G\Lambda}{\nu - \Lambda} \right) + \mathcal{O}(1). \]

\[ 2n\nu \gg 1 \]

\[ \sigma_N = 2\mathcal{N}^2 + \mathcal{O}(\mathcal{N}) \quad \text{if} \quad \mathcal{N} \gg 1. \]

\[ \nu \approx \left| \int_0^T \chi(t) \exp(-2i\omega_0 t) \, dt \right| \]

\[ \omega(t) \equiv \omega_0 [1 + \chi(t)] \]

\[ \Lambda = \int_{t_i}^{t_f} \gamma(\tau) \, d\tau \]

\[ n^2 \nu \ll 1 \]
\[ \nu = \sqrt{|g|^2 - \delta^2} \]

\[ \delta = \omega_0 (T - T_{res}) \]

\[ |g| \ll 1 \text{ and } |\delta| \ll 1 \]

The amplification can be observed if \(|\delta| < |g|\),

\[ |w - w_{res}| < \delta w_{res} = \frac{2}{\pi} |g| w_0 \]

\[ w_{res} = 2w_0 \left(1 - \varphi / \pi \right) \]

\[ T_{res} = \frac{1}{2} T_0 \left(1 + \varphi / \pi \right) \]

\[ \varphi = -\omega_0 \int_{t_i}^{t_f} \chi(t) dt \]
\[ f(m) \approx \frac{\exp\left[-\left(m + \frac{1}{2}\right)/(2N)\right]}{\sqrt{2\pi N(m + 1/2)}} \]

\[ f(m) \equiv \langle m|\hat{\rho}|m\rangle \]

\( N = 1000 \)
Squeezing

For a single mode with $\omega(t) = \omega_0 [1 + 2 \varepsilon \sin (2 \omega_0 t)]$, $|\varepsilon| \ll 1$, and the initial vacuum state:

$$U = \frac{1}{2} e^{-2\tau}, \quad V = \frac{1}{2} e^{2\tau}, \quad \tau = \varepsilon \omega_0 t \quad [\text{or} \quad \tau = n\nu]$$

$$U = \langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2, \quad V = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2$$

The photon distribution function strongly oscillates:

$$f(2m) = \frac{[\tanh(\tau)]^{2m}}{\cosh(\tau)} \frac{(2m)!}{(2^m m!)^2}, \quad f(2m + 1) = 0.$$

For coupled modes the squeezing coefficient depends on the ``competition” between the dimensionless energy $\tilde{\mathcal{E}} \equiv \mathcal{E}/\hbar \omega_0$ and the invariant uncertainty product

$$\mathcal{D} \equiv \langle \tilde{q}^2 \rangle \langle \tilde{p}^2 \rangle - \langle (\tilde{q}\tilde{p} + \tilde{p}\tilde{q})/2 \rangle^2, \quad \tilde{q} = q - \langle q \rangle$$

The universal invariant for a single mode

$$s = \frac{2\mathcal{D}}{\tilde{\mathcal{E}} + \sqrt{\tilde{\mathcal{E}}^2 - \mathcal{D}}}$$
an explicit correlation between the existence of squeezing ($S < 1$) and the oscillations of the PDF is observed
\[ f(m) \approx \left[ \pi \tau (m + 1/2) \right]^{-1/2} \{ \exp \left[ -(m + 1/2)/\tau \right] \\
+ (-1)^m \exp \left[ -(m + 1/2)G_0^2/\tau \right] \}. \]

\[ \tau \approx 2N \gg 1, \ 1 \ll m \ll \tau^2 \text{ and } G\Lambda/\nu \ll 1. \]
Figure 1. The relative shift of the cavity eigenfrequency $\chi$ (upper curves) and normalized damping coefficient $\gamma$ (lower curves) versus the dimensionless time variable $\zeta = \beta_1 t = \tau/Z$ for $A_0 = 10$ and fixed values of the parameters $g$ and $h$ (from right to left): $g = 0$ and $h$ arbitrary; $g = 10$ and $h = 1$; $g = 10$ and $h = 10$; $g = 10$ and $h = 100$. 
Figure 2. The dependence of the amplification coefficient $F$ on the parameter $Z$ for fixed values of the parameter $A_0 = 4, 6, 10, 20$ (from bottom to top; in the case of $h = 100$ the plots are made for $A_0 = 4, 10, 20$). The plots with $g = 0$ do not depend on $h$. In all plots with fixed values of $h$ we use the value $g = 10$. 
\( g = 0 \)
Classical field amplification in the re-entrant cavity with periodically illuminated semiconductor slabs:

For initial coherent state $|\alpha\rangle$ with $\alpha = |\alpha| \exp(i\theta)$ and $|\alpha| \gg 1$:

$$\mathcal{N}_n^{(\alpha)} = |\alpha|^2 e^{-2n\Lambda} \left| \cosh(n\nu) - \sinh(n\nu) e^{i(\varphi+2\theta)} \right|^2$$

Phase averaged value:

$$\overline{\mathcal{N}_n^{(\alpha)}} = \frac{1}{2} |\alpha|^2 e^{2n(\nu-\Lambda)}$$
For the initial classical RF-signal of the form

\[ E_{in}(t) = E_0 \cos(\omega_0 t - \vartheta) \]

the field component after \( n \) pulses started at \( t = 0 \)

\[ E_n(t) = D_n(t) \text{Re} (\varepsilon(t)e^{i\vartheta}) \]

\[ \ddot{\varepsilon} + \omega^2(t)\varepsilon = 0 \quad \varepsilon(t < 0) = \exp(-i\omega_0 t) \]

\[ D_n(t) = E_0 \exp \left[ -n \left( \Lambda + \frac{\pi}{2Q} \right) - \alpha(t - nT') \right] \]

\[ Q = \omega_0/(2\alpha) \]
\[ E_n(t) = D_n(t) \left\{ \cosh(n\nu) \cos[\psi_n(t) - \vartheta] - \frac{\delta}{\nu} \sinh(n\nu) \sin[\psi_n(t) - \vartheta] \\
+ \frac{|g|}{\nu} \sinh(n\nu) \cos[\psi_n(t) + \vartheta + \phi + \omega_0 \Delta] \right\} \]

\[ \psi_n(t) = \omega_0 t - n \omega_0 \Delta \]
\[ \Delta = T - T_0/2 \]

\[ \nu = \sqrt{|g|^2 - \delta^2} \]

\[ \delta = \omega_0 (T - T_{res}) \]

\[ |g| \ll 1 \text{ and } |\delta| \ll 1 \]

The amplification can be observed if \(|\delta| < |g|\),

\[ |w - w_{res}| < \delta w_{res} = \frac{2}{\pi} |g| w_0 \]

\[ T_{res} = \frac{1}{2} T_0 \left(1 + \varphi/\pi\right) \]

\[ w_{res} = 2w_0 \left(1 - \varphi/\pi\right) \]

\[ \varphi = -\omega_0 \int_{t_i}^{t_f} \chi(t) dt \]
The amplitude of these oscillations equals

\[ A_n = D_n(t) \left\{ 1 + \frac{2|g|^2}{\nu^2} \sinh^2(n\nu) \left[ 1 + \frac{\delta}{|g|} \sin(\Psi) \right] + \frac{|g|}{\nu} \sinh(2n\nu) \cos(\Psi) \right\}^{1/2} \]

\[ \Psi = 2\vartheta + \phi + \omega_0 \Delta \]

Depending on the phase \( \Psi \), this amplitude can assume the values between

\[ A_n^{(\pm)} = D_n(t) \left\{ 1 + \frac{2|g|^2}{\nu^2} \sinh^2(n\nu) \pm \frac{|g|}{\nu} \sinh(2n\nu) \sqrt{1 + \frac{\delta^2}{\nu^2} \tanh^2(n\nu)} \right\}^{1/2} \]

In the case of strict resonance, \( \delta = 0 \) and \( \nu = |g| \),

\[ A_n^{(\pm)} = D_n(t)e^{\pm|g|n} \]
\[ \langle A_n \rangle_{\Psi} = \int_{0}^{2\pi} A_n(\Psi) \frac{d\Psi}{2\pi} = \frac{2}{\pi} D_n q(x) E(k(x)) \]

\[ E(k) = \int_{0}^{\pi/2} \sqrt{1 - k^2 \sin^2(\varphi)} \, d\varphi \]

\[ x = \frac{\delta}{|g|} = \pi \frac{w_{res} - w}{2w_0|g|} \]

\[ q(x) = \frac{\sinh(b\sqrt{1-x^2}) + \sqrt{\cosh^2(b\sqrt{1-x^2}) - x^2}}{\sqrt{1-x^2}} \]

\[ k(x) = \frac{2\sqrt{\sinh(b\sqrt{1-x^2}) \sqrt{\cosh^2(b\sqrt{1-x^2}) - x^2}}}{\sinh(b\sqrt{1-x^2}) + \sqrt{\cosh^2(b\sqrt{1-x^2}) - x^2}} \]

\[ b = |g|n \]
$$a(x) = \frac{\langle A_n \rangle \Psi(x)}{\langle A_n \rangle \Psi(0)} = \frac{e^{-b} q(x) E(k(x))}{E\left(\sqrt{1-e^{-4b}}\right)}$$

$$a(\infty) = \frac{\pi e^{-b}}{2E\left(\sqrt{1-e^{-4b}}\right)}$$

$$s(x) = \frac{\sqrt{\langle A_n^2 \rangle \Psi(x) - [\langle A_n \rangle \Psi(x)]^2}}{\langle A_n \rangle \Psi(0)}$$

$$= B \sqrt{\frac{\pi^2}{4} \cosh\left(\frac{2b\sqrt{1-x^2}}{1-x^2}\right) - x^2} - q^2(x) E^2(k(x)).$$

for $b > 1$

$$s_{\text{theor}}(0) \approx \sqrt{\frac{\pi^2}{8} - 1} \approx 0.48$$

$$s(x_1 + \eta) = |\eta| e^{-b} \frac{b^2}{2 \sqrt{2}} \sqrt{(b/\pi)^2 + 1},$$

$$b \sqrt{\frac{x_m^2}{m} - 1} = m \pi$$
If the cavity is not illuminated, the amplitude of the classical signal decays as

\[ A_{\text{free}}(t) = E_0 \exp(-\alpha t) \quad \Rightarrow \quad A_{\text{free}}^n = E_0 \exp(-n\pi/2Q) \]

\[ C_n(x) \equiv \frac{\langle A_n \rangle_\Psi(x)}{A_{\text{free}}^n} = \frac{2}{\pi} e^{-n\Lambda} q(x) E(k(x)) \]

\[ A_n^{\text{max}}(0) = (\pi/2) \langle A_n \rangle_\Psi(0) \]

\[ C_n(0) = \frac{2}{\pi} e^{-n(\Lambda-|g|)} \]

\[ F = \frac{1}{n} \ln[C_n(0)\pi/2] \]

\[ F = |g| - \Lambda \]
Interesting theoretical problems

Interaction between the field and detectors


\[ H = \frac{1}{2} \left[ P^2 + \omega^2(t)Q^2 + p^2 + \omega_0^2q^2 - 4\omega_0\eta pQ \right] \]

\( p, q \) - probe oscillator

\( P, Q \) - field oscillator

If \( \varepsilon \ll \eta \ll 1 \), the field mode is in a mixed state:

\[ \text{Tr} \dot{\hat{\rho}}^2 = \left[ \cosh(\tau) \right]^{-1} \]

\[ \mathcal{N} = \frac{1}{2} \sinh^2(\tau) \]

\[ U_{\min} = \frac{1}{4} \left( 1 + e^{-2\tau} \right) \]

\[ f(2m) = \frac{[\tanh(\tau)]^{2m}}{\cosh(\tau)} \frac{(2m)!}{(2^m m!)^2}, \quad f(2m + 1) = 0. \]

Squeezed vacuum

In the presence of detector

\[ f(n) = \frac{2(i\bar{z})^n P_n(-i\bar{z})}{\sqrt{1 + 3 \cosh^2 \tau}}, \quad \bar{z} = \frac{\sinh \tau}{\sqrt{1 + 3 \cosh^2 \tau}} \]
Important unsolved theoretical problems:

• Numerical calculation of the frequency shift and EM field inside the re-entrant cavity

• Interaction between the field and detectors

• Full quantization scheme, taking into account dissipation in media with time-dependent parameters and moving boundaries