Dynamical Casimir effect: A global view

Iwo Bialynicki-Birula

Center for Theoretical Physics, Warsaw

Padova June 2011

Outline

- **Early history**
- **Quantum mechanics of a simple harmonic oscillator**
- **Dynamical Casimir effect of the harmonic oscillator**
- **The EM field: a huge oscillator**
- **The ground state and squeezed states**
- **Dynamical Casimir effect of the EM field**

The same physics under three different names

- **Dynamical Casimir effect**
- **Parametric pair creation**
- **Generation of squeezed states**

It all began with Schrödinger in 1939

In his paper "The proper vibrations of the expanding universe" Physica 6, 899 (1939) Schrödinger wrote: "These two proper vibrations cannot be rigorously separated in the expanding universe... Generally speaking this is a phenomenon of outstanding importance. With particles it would mean production or annihilation of matter, merely by expansion...

Alarmed by these prospects, I have examined the matter in more detail...the alarming phenomena (i.e. pair production and the reflection of light in space)... would probably be caused by an accelerated expansion"

Original Casimir effect

In 1948 Hendrik G. B. Casimir discovered theoretically that two parallel conducting plates placed at the distance d attract each other with the force per unit area equal to

$$f = -\frac{\pi^2 \hbar c}{240 d^4}$$

This force is extremely weak $0.0013 N/m^2$ for $d = 1$micron
It took half a century to confirm experimentally the Casimir prediction (with the use of a torsion pendulum)
S. K. Lamoreaux, Demonstration of the Casimir force in the 0.6 to 6μm range, Phys. Rev. Lett. 78, 5 (1997)

Twenty years later

Modern history of the dynamical Casimir effect begins in 1970 with the work of G. T. Moore
"Quantum theory of the electromagnetic field in a variable-length one-dimensional cavity"
Starting with this paper there was a flurry of activity
Many papers appeared on the phenomena caused by moving boundaries

I believe that we were the first to depart from moving boundaries and link photon pair production directly with time dependent medium and squeezed states of light
Our paper "Space-time description of squeezing" published in JOSA B 4, 1621 (1987) has barely been noticed
The name dynamical Casimir effect was introduced by Yablonovitch (1989) and Schwinger (1992)

Quantum mechanics of an oscillator

Time-independent Hamiltonian in n dimensions

$$H_0 = \frac{1}{2}\left(p_i g^{ij} p_j + x^i u_{ij} x^j\right)$$

The wave function of the initial ground state in the coordinate representation

$$\psi_0(\boldsymbol{r}, t) = C \exp\left(-\frac{1}{2} x^i \Omega_{ij} x^j\right) \exp\left(-iE_0 t\right)$$

Atomic units are used ($\hbar = 1 = m$)
Ω is the symmetric positive matrix satisfying the following matrix equation

$$\hat{\Omega}\hat{g}\hat{\Omega} = \hat{u} \qquad \Omega \approx \sqrt{u/g}$$

Gaussian states

Theorem: Every Gaussian state will remain Gaussian if the time-evolution is governed by a (possibly time-dependent) quadratic Hamiltonian

$$H(t) = \frac{1}{2}\left(p_i g^{ij}(t) p_j + x^i u_{ij}(t) x^j\right) - f_i(t) x^i$$

The general form of the wave function of a Gaussian state in coordinate and momentum representations is

$$\psi(\boldsymbol{r}, t) = C_x(t) \exp\left[-\tfrac{1}{2}(x^i - \xi^i(t)) K_{ij}(t)(x^j - \xi^j(t)) + i x^i \pi_i(t)\right]$$

$$\tilde{\psi}(\boldsymbol{p}, t) = C_p(t) \exp\left[-\tfrac{1}{2}(p_i - \pi_i(t)) L^{ij}(t)(p_j - \pi_j(t)) - i p_i \xi^i(t)\right]$$

where $\hat{L}(t) = \hat{K}^{-1}(t)$

Time evolution of Gaussian states

The Schrödinger equation for $\psi(\boldsymbol{r}, t)$

$$i\partial_t \psi(\boldsymbol{r}, t) = H(t)\psi(\boldsymbol{r}, t)$$

leads to the matrix Riccati equation for $\hat{K}$ and $\hat{L}$

$$i\frac{d}{dt}\hat{K}(t) = \hat{K}(t)\,\hat{g}(t)\,\hat{K}(t) - \hat{u}(t) \qquad i\frac{d}{dt}\hat{L}(t) = \hat{L}(t)\,\hat{u}(t)\,\hat{L}(t) - \hat{g}(t)$$

and the standard mechanical equations for $\xi(t)$ and $\pi(t)$

$$\frac{d}{dt}\xi^i(t) = g^{ij}(t)\pi_j(t) \qquad \frac{d}{dt}\pi_i(t) = -u_{ij}(t)\xi^j(t) + f_i(t)$$

Two mechanisms of excitation

Two special families of Gaussian states are:
Pure coherent states when $\hat{K} = \hat{\Omega}$ and (ξ, π) do not vanish
Pure squeezed states when $\hat{K} \neq \hat{\Omega}$ and (ξ, π) do vanish

These two families are generated by two different mechanisms:

Coherent states are generated by an application of external forces $f_i(t)$
Squeezed states are generated by time-dependent parameters $\hat{g}(t)$ and $\hat{u}(t)$
The dynamical Casimir effect is associated with the second mechanism

Second mechanism has a significant quantum component

The external force has the same effect
in classical and in quantum case
It will either displace the trajectory of the classical particle
or of the center of the quantum wave packet
In contrast, time-dependent parameters
may produce a purely quantum effect

Assume that a classical particle is initially at rest
No amount of shaking could start it moving
On the other hand, the ground state of
a quantum particle will change
It will become a combination of excited states

Wigner function $W(\boldsymbol{r}, \boldsymbol{p}, t)$

$$W(\boldsymbol{r}, \boldsymbol{p}, t) = \int d^n\eta \exp(i\boldsymbol{\eta}\cdot\boldsymbol{p}/\hbar)\psi(\boldsymbol{r} - \boldsymbol{\eta}/2, t)\psi^*(\boldsymbol{r} + \boldsymbol{\eta}/2, t)$$

Evolution equation for W

$$\partial_t W(\boldsymbol{r}, \boldsymbol{p}, t) = -\left(p_i g^{ij}(t)\frac{\partial}{\partial x^j} - x^i u_{ij}(t)\frac{\partial}{\partial p_j}\right) W(\boldsymbol{r}, \boldsymbol{p}, t)$$

Theorem: General solution of the initial value problem for the classical harmonic oscillator gives the exact solution for the quantum Wigner function

$$W(\boldsymbol{r}, \boldsymbol{p}, t) = W(\boldsymbol{\xi}(\boldsymbol{r}, \boldsymbol{p}, -t), \boldsymbol{\pi}(\boldsymbol{r}, \boldsymbol{p}, -t), 0)$$

The arguments $\boldsymbol{r}$ and $\boldsymbol{p}$ of W are the classical initial data

Wigner function of the ground state

Evaluation of Gaussian integrals
for the ground state wave function gives

$$W_0(\boldsymbol{r},\boldsymbol{p}) = \exp\left[-\frac{1}{\hbar^2}\left(p_i(\Omega^{-1})^{ij}p_j + x^i\Omega_{ij}x^j\right)\right]$$

It will be shown later that the exponent
has an interesting interpretation

Wigner function for any Gaussian state

Long calculations result in a remarkably compact formula

$$W(\boldsymbol{r}, \boldsymbol{p}, t) = \exp[-\frac{2}{\hbar^2}\langle \hat{S}(t)\hat{S}(t)\rangle]$$

Constant normalization factor has been omitted

$$\hat{S}(t) = \Delta x^i(t)\Delta\hat{p}_i(t) - \Delta p_i(t)\Delta\hat{x}^i(t)$$

Displacements from the classical trajectories

$$\Delta x^i(t) = x - \xi^i(t) \qquad \Delta p_i(t) = p_i - \pi^i(t)$$

$$\Delta\hat{x}^i(t) = \hat{x}^i(t) - \xi^i(t) \qquad \Delta\hat{p}_i(t) = \hat{p}_i(t) - \pi^i(t)$$

All operators are evaluated in the Heisenberg picture

the expectation valuesClassical trajectories are equal to

$$\langle \hat{x}^i(t)\rangle = \xi^i(t) \quad \langle \hat{p}_i(t)\rangle = \pi^i(t)$$

Evolution of the initial ground state

Heisenberg operators depend linearly on their initial data

For the ground state $\xi^i(t) = 0$ and $\pi^i(t) = 0$

$$\hat{\boldsymbol{r}}(t) = \hat{\alpha}(t)\cdot\hat{\boldsymbol{r}} + \hat{\beta}(t)\cdot\hat{\boldsymbol{p}} \quad \hat{\boldsymbol{p}}(t) = \hat{\gamma}(t)\cdot\hat{\boldsymbol{r}} + \hat{\delta}(t)\cdot\hat{\boldsymbol{p}}$$

$$\begin{aligned} \hat{S}(t) &= \boldsymbol{r}\cdot\hat{\boldsymbol{p}}(t) - \boldsymbol{p}\cdot\hat{\boldsymbol{r}}(t) = \boldsymbol{\xi}(\boldsymbol{r},\boldsymbol{p},-t)\cdot\hat{\boldsymbol{p}} - \boldsymbol{\pi}(\boldsymbol{r},\boldsymbol{p},-t)\cdot\hat{\boldsymbol{r}} \\ &= \boldsymbol{r}\cdot\left[\hat{\gamma}(t)\cdot\hat{\boldsymbol{r}} + \hat{D}(t)\cdot\hat{\boldsymbol{p}}\right] - \boldsymbol{p}\cdot\left[\hat{\alpha}(t)\cdot\hat{\boldsymbol{r}} + \hat{\beta}(t)\cdot\hat{\boldsymbol{p}}\right] \end{aligned}$$

To evaluate $\langle\hat{S}(t)\hat{S}(t)\rangle$ for the initial ground state we only need

$$\langle\hat{x}^i\hat{x}^j\rangle = \frac{\hbar}{2}(\Omega^{-1})^{ij} \quad \langle\hat{p}_i\hat{p}_j\rangle = \frac{\hbar}{2}\Omega_{ij} \quad \langle\hat{x}^i\hat{p}_j + \hat{p}_j\hat{x}^i\rangle = 0$$

$$2\langle\hat{S}(t)\hat{S}(t)\rangle = \hbar\,[\boldsymbol{r},\ \boldsymbol{p}]\begin{bmatrix} \gamma(t) & \hat{\delta}(t) \\ \hat{\alpha}(t) & \beta(t) \end{bmatrix}\begin{bmatrix} \Omega & 0 \\ 0 & \Omega^{-1} \end{bmatrix}\begin{bmatrix} \hat{\gamma}^T(t) & \hat{\alpha}^T(t) \\ \hat{\delta}^T(t) & \hat{\beta}(t)^T \end{bmatrix}\begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{p} \end{bmatrix}$$

Particle production

The number of quanta for the harmonic oscillator in 1D

$$\hat{N} = \frac{\hat{H}}{\hbar\omega} = \frac{\hat{p}^2}{2\hbar\omega} + \frac{\omega\hat{x}^2}{2\hbar}$$

Generalization to n dimensions

$$\hat{N} = \frac{1}{2\hbar}\left(\hat{p}_i(\Omega^{-1})^{ij}\hat{p}_j + \hat{x}^i\Omega_{ij}\hat{x}^j\right)$$

The expectation value of N evaluated in the the state described by our Wigner function is given by the integral

$$\langle\hat{N}\rangle_t = \frac{1}{2}\int d^n r \int d^n p\, W(\boldsymbol{r},\boldsymbol{p},t)\left(p_i(\Omega^{-1})^{ij}p_j + x^i\Omega_{ij}x^j\right)$$

$$= \frac{1}{4}\mathrm{Tr}\left\{\begin{bmatrix}\gamma(t) & \hat{\delta}(t)\\ \hat{\alpha}(t) & \beta(t)\end{bmatrix}\begin{bmatrix}\Omega & 0\\ 0 & \Omega^{-1}\end{bmatrix}\begin{bmatrix}\hat{\gamma}^T(t) & \hat{\alpha}^T(t)\\ \hat{\delta}^T(t) & \hat{\beta}(t)^T\end{bmatrix}\begin{bmatrix}\Omega & 0\\ 0 & \Omega^{-1}\end{bmatrix}\right\}$$

Dynamical Casimir effect for an oscillator

An intrinsically quantum process of particle production was fully described in terms of solutions of classical equations of motion by going through the following steps

- **Find the frequency matrix Ω for the ground state**
- **Solve the general initial value problem for the oscillator**
- **Evaluate the projection of the evolved ground state on the initial ground state**

Quantized electromagnetic field

The canonical variables $\boldsymbol{r} \rightarrow \boldsymbol{A}$ and $\boldsymbol{p} \rightarrow -\boldsymbol{D}$

$$\hat{\boldsymbol{A}}(\boldsymbol{r})\Psi\left[\boldsymbol{A}\right] = \boldsymbol{A}(\boldsymbol{r})\Psi\left[\boldsymbol{A}\right] \qquad \hat{\boldsymbol{D}}(\boldsymbol{r})\Psi\left[\boldsymbol{A}\right] = i\hbar\frac{\delta}{\delta\boldsymbol{A}(\boldsymbol{r})}\Psi\left[\boldsymbol{A}\right]$$

The Hamiltonian

$$\hat{H}(t) = \frac{1}{2}\int d^3r\,\Big[-\frac{\delta}{\delta\boldsymbol{A}(\boldsymbol{r})}\frac{\hbar^2}{\epsilon(\boldsymbol{r},t)}\frac{\delta}{\delta\boldsymbol{A}(\boldsymbol{r})} + \big(\nabla\times\boldsymbol{A}(\boldsymbol{r})\big)\frac{1}{\mu(\boldsymbol{r},t)}\big(\nabla\times\boldsymbol{A}(\boldsymbol{r})\big)\Big]$$

Ground state functional (arbitrary normalization)

$$\Psi_0\left[\boldsymbol{A}\right] = \exp\left[-\frac{1}{4\pi^2\hbar}\sqrt{\frac{\epsilon}{\mu}}\int d^3r\int d^3r'\,(\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r}))\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|^2}(\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r}'))\right]$$

The electromagnetic Wigner function

$$W[\boldsymbol{A},\boldsymbol{D},t] = \exp[-\frac{2}{\hbar^2}\langle\hat{S}(t)\hat{S}(t)\rangle]$$

$$\hat{S}(t) = \int d^3r \left[\boldsymbol{A}(\boldsymbol{r})\cdot\hat{D}(\boldsymbol{r},t) - \boldsymbol{D}(\boldsymbol{r})\cdot\hat{A}(\boldsymbol{r},t)\right]$$

Ground state Wigner function

$$W[\boldsymbol{A},\boldsymbol{D}] = \exp\left(-2N\left[\boldsymbol{B},\boldsymbol{D}\right]\right)$$

The exponent is twice the number of photons

The formula for $N[\boldsymbol{B},\boldsymbol{D}]$ was given by Zeldovich in 1965

$$2N\left[\boldsymbol{B},\boldsymbol{D}\right] = \frac{1}{2\pi^2\hbar}\int d^3r\int d^3r'$$

$$\times\left(\sqrt{\frac{\mu}{\epsilon}}\frac{\boldsymbol{D}(\boldsymbol{r})\cdot\boldsymbol{D}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^2} + \sqrt{\frac{\epsilon}{\mu}}\frac{(\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r}))\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r}'))}{|\boldsymbol{r}-\boldsymbol{r}'|^2}\right)$$

Photon pair production

In order to follow the same path as in quantum mechanics of a harmonic oscillator we must solve the initial value problem for the Maxwell equations

Exact analytic solutions of Maxwell equations even for some simple time dependence of $\epsilon(\boldsymbol{r},t)$ and $\mu(\boldsymbol{r},t)$ are hard to find

One can do it when there is no dependence on $\boldsymbol{r}$

I. Bialynicki-Birula and Z. Bialynicka-Birula

Phys. Rev. A 78, 042109 (2008)

I. Bialynicki-Birula and Ł. Rudnicki

Opt. Comm. 283, 644 (2010)

In other cases one has to resort to perturbation theory

I. Bialynicki-Birula and Z. Bialynicka-Birula

Phys. Rev. A 77, 052103 (2008)

Back to Schrödinger

Schrödinger considered expanding Universe but
Every time-varying metric produces photon pairs
Field equations in any metric have the same Maxwellian form

$$\partial_t \boldsymbol{D} = \boldsymbol{\nabla} \times \boldsymbol{H}$$
$$\partial_t \boldsymbol{B} = -\boldsymbol{\nabla} \times \boldsymbol{E}$$

The constitutive relations are in general anisotropic

$$\boldsymbol{E} = \hat{M}\boldsymbol{D} - \hat{N}\boldsymbol{B} \quad \boldsymbol{H} = \hat{M}\boldsymbol{B} + \hat{N}\boldsymbol{D}$$

$$M_{ij} = -\frac{g_{ij}}{g^{00}\sqrt{-g}} \quad N_{ij} = \frac{g^{0k}\epsilon_{ikj}}{g^{00}}$$

Time dependence of $g_{\mu\nu}$ has the same effect
as time-dependent ϵ or μ: Production of photon pairs

Summary

- **Dynamical Casimir effect has a quantum origin but its mathematical description requires only classical analysis**
- **Time variation of the medium is always accompanied by the production of photon pairs**
- **Nonrelativistic quantum mechanics can be applied to the electromagnetic field treated as one huge oscillator**
- **Wigner function with its direct connection to the photon number is a very useful tool to study pair production in the dynamical Casimir effect**